Nonlinear Anal. RWA 84 (2025) 104327

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

Traveling wave in a ratio-dependent Holling–Tanner system with nonlocal diffusion and strong Allee effect[☆]

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ARTICLE INFO

Keywords:
Holling–Tanner system
Traveling waves
Upper and lower solutions
Strong Allee effect

ABSTRACT

This paper explores a ratio-dependent Holling–Tanner predator–prey system with nonlocal diffusion, wherein the prey is subject to strong Allee effect. To be specific, by using Schauder's fixed point theorem and iterative technique, we establish a theoretical framework regarding the existence of traveling waves. We meticulously construct upper and lower solutions and a novel sequence, and employ the squeeze method to validate the existence of traveling waves for $c > c^*$. Additionally, by spreading speed theory and the comparison principle, we confirm the existence of traveling wave with $c = c^*$. Finally, we investigate the nonexistence of traveling waves for $c < c^*$, and conclusively determine the minimal wave speed.

1. Introduction

Predation, as one of the fundamental interactions in nature, plays a significant role in shaping ecosystems. However, the classical Lotka–Volterra predator–prey system might be inadequate for depicting the dynamics of ecological systems, as populations in reality face limiting factors like resource competition and diseases, which are not fully considered in such a model. In addition, predators can adjust their hunting strategies according to the changes in both the density and availability of the prey. Various ecological factors also influence the dynamics between the predator and the prey, such as habitat fragmentation, climate change and human activities. Hence, many scholars have dedicated themselves to investigating the complex ecological phenomena in predator–prey systems, seeing [1–3].

Predator-prey reaction–diffusion systems are motivated from the spatial heterogeneity observed in natural habitats. The Holling–Tanner system with Holling-II type functional response

$$\begin{cases} u_t = d_1 \Delta u + u(1-u) - \dfrac{auv}{1+u}, \\ v_t = d_2 \Delta v + sv\left(1 - \dfrac{v}{u}\right) \end{cases}$$

☆ This work is supported by the National Natural Science Foundation of China (No. 12171039 and 12271044) and the Natural Science Foundation of Tianjin (No. 23JCQNJC01010).

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https://doi.org/10.1016/j.nonrwa.2025.104327
Received 4 January 2024; Received in revised form 22 January 2025; Accepted 24 January 2025
Available online 6 February 2025
1468-1218/

has recently attracted increasing interest [4,5]. According to [6], the ratio-dependent Holling–Tanner system provides a way to avoid the "biological control paradox" wherein classical prey-dependent exploitation models generally fail to achieve a low and stable pest prey equilibrium density. Aligning with perspectives presented in [7,8], the functional responses on ecological timescales should depend on both prey and predator densities, with an emphasis on their ratio. Hence, the authors in [7] proposed a ratio-dependent functional response

$$f\left(\frac{u}{v}\right)=\frac{m\left(\frac{u}{v}\right)}{\left(\frac{u}{v}\right)+a}=\frac{mu}{u+av},$$

and the corresponding Holling–Tanner system is studied in [9]. Traveling waves of the Holling–Tanner system have been further studied to refine the understanding of population propagation dynamics, we refer readers to [10–16] for local diffusion systems, and [17,18] for nonlocal diffusion systems.

In the aforementioned Holling–Tanner systems, traveling waves are considered when the growth of the prey is modeled by a logistic growth pattern. The Allee effect has garnered widespread attention in recent years owing to its complex nature and practical significance [19–22], it refers to reduced fitness or decline in population growth at low population densities, this scenario poses challenges in finding mates, resulting in declining birth rates and increasing the risk of extinction [23]. This phenomenon can be categorized into two types: weak Allee effect and strong Allee effect [24,25]. The former refers to a scenario where the population has a positive and increasing growth when the size of the population is below a certain threshold value, and the latter pertains to a situation where growth is negative when the size of the population is below a certain threshold value.

When the prey is affected by the Allee effect, the main challenge lies in skillfully constructing appropriate upper and lower solutions to establish the existence of traveling waves. Recently, Zhao and Wu in [14] investigated the existence of traveling waves of the Holling–Tanner system with Lotka–Volterra functional response and strong Allee effect

$$\begin{cases} u_t = d_1 \Delta u + u(1-u)\left(\frac{u}{b}-1\right) - auv, \\ v_t = d_2 \Delta v + sv\left(1-\frac{v}{u}\right). \end{cases} \tag{1}$$

To the best of the authors' knowledge, there has been limited research into the existence of traveling waves of the ratio-dependent Holling–Tanner systems with nonlocal diffusion and strong Allee effect. Consequently, this work investigates the following system

$$\begin{cases} u_t = d_1 \mathcal{N}_1[u](x,t) + u(1-u)\left(\frac{u}{b}-1\right) - \frac{muv}{u+av}, \\ v_t = d_2 \mathcal{N}_2[v](x,t) + sv\left(1-\frac{v}{u}\right), \end{cases} \tag{2}$$

where $u(x,t)$ and $v(x,t)$ stand for the population densities of the prey and the predator, respectively. d_1 and d_2 are the diffusion coefficients of the prey and the predator; $b \in (0,1)$ represents the Allee threshold value; m is a measure of the quality of the prey as food for the predator; a and s denote the saturation rate of the predator and a measure of the growth rate of the predator. And all parameters are positive. Moreover, $\mathcal{N}_i[w](x,t)$, $i=1,2$, formulate the spatial nonlocal diffusion of individuals

$$\mathcal{N}_i[w](x,t) = \int_{\mathbb{R}} J_i(x-y)w(y,t)dy - w(x,t),$$

where the kernel functions $J_i : \mathbb{R} \to \mathbb{R}$, $i=1,2$, satisfy

(J1) $J_i \in C^1(\mathbb{R})$, $J_i(x) = J_i(-x) \geq 0$ and $\int_{\mathbb{R}} J_i(x)dx = 1$.

(J2) J_i satisfy the decay bounds:

$$\int_{\mathbb{R}} J_i(x)e^{\lambda x}dx < +\infty \text{ for any } \lambda \in (0,\lambda_0) \text{ and } \lim_{\lambda\to\lambda_0}\int_{\mathbb{R}} J_i(x)e^{\lambda x}dx = +\infty$$

for some $\lambda_0 \in (0,+\infty]$ and $\int_{\mathbb{R}} |J_i'(x)|dx < +\infty$.

Let us consider (3) without diffusion, that is

$$\begin{cases} u'(t) = u(1-u)\left(\frac{u}{b}-1\right) - \frac{muv}{u+av}, \\ v'(t) = sv\left(1-\frac{v}{u}\right). \end{cases} \tag{3}$$

Obviously, $(b,0)$ and $(1,0)$ are nonnegative equilibria of (3). Denote

$$b_1 = 1 + \frac{2m}{1+a} - 2\sqrt{\frac{m}{1+a}\left(1+\frac{m}{1+a}\right)}.$$

It is simple to check that $b_1 \in (0,1)$ and

(1) if $0 < b < b_1$, then (3) has two positive equilibria (u_1^*, u_1^*) and (u_2^*, u_2^*) with $u_1^* < u_2^* < 1$,

(2) if $b = b_1$, then (3) has a unique positive equilibrium $((1+b_1)/2, (1+b_1)/2)$,

(3) if $b_1 < b < 1$, then (3) has no positive equilibrium,

where

$$u_1^* = \frac{1}{2}\left(b+1-\sqrt{b^2-2\left(1+\frac{2m}{1+a}\right)b+1}\right),$$
$$u_2^* = \frac{1}{2}\left(b+1+\sqrt{b^2-2\left(1+\frac{2m}{1+a}\right)b+1}\right).$$

Our primary goal is to establish the existence of traveling waves connecting the predator-free state and the coexistence state in (2). Let us first assume $0 < b < b_1$, which is equivalent to $4mb < (1-b)^2(1+a)$ for $b \in (0,1)$. Referencing [26], at this point, the positive equilibrium (u_1^*, u_1^*) is always unstable, whereas (u_2^*, u_2^*) can be stable under certain conditions. Therefore, our attention is directed towards identifying traveling waves connecting $(1,0)$ and (u_2^*, u_2^*). In what follows, it will be convenient to use the following notations

$$f(\phi,\psi) = (1-\phi)\left(\frac{\phi}{b}-1\right) - \frac{m\psi}{\phi+a\psi}, \quad g(\phi,\psi) = s\left(1-\frac{\psi}{\phi}\right).$$

A positive solution is called a traveling wave, if it has the form

$$(u,v)(x,t) = (\phi,\psi)(\xi), \ \ \xi = x+ct,$$

where $c > 0$ is the wave speed. Then $(\phi,\psi)(\xi)$ satisfies

$$\begin{cases} c\phi'(\xi) = d_1\mathcal{N}_1[\phi](\xi) + \phi(\xi)f(\phi,\psi)(\xi), \\ c\psi'(\xi) = d_2\mathcal{N}_2[\psi](\xi) + \psi(\xi)g(\phi,\psi)(\xi), \end{cases} \tag{4}$$

where

$$\mathcal{N}_i\,[w]\,(\xi) = \int_{\mathbb{R}} J_i(\xi-y)w(y)dy - w(\xi), \ \ i=1,2.$$

If $(\phi,\psi)(\xi)$ further meets the boundary conditions

$$(\phi,\psi)(-\infty) = (1,0) \ \text{ and } \ (\phi,\psi)(+\infty) = (u_2^*, u_2^*), \tag{5}$$

then it is called a traveling wave connecting $(1,0)$ and (u_2^*, u_2^*), also named a invasion wave [27]. Therefore, establishing the existence of traveling waves of (2) is precisely equivalent to demonstrating the existence of the positive solutions of the boundary value problem (4)–(5).

In order to describe our main results, let

$$\Delta(\lambda,c) = d_2\left(\int_{\mathbb{R}} J_2(y)e^{-\lambda y}dy - 1\right) - c\lambda + s.$$

Through a straightforward calculation, it is easy to get

$$\begin{aligned}
&\Delta(0,c) = s > 0 \text{ and } \lim\nolimits_{\lambda\to+\infty}\Delta(\lambda,c) = +\infty \text{ for all } c,\\
&\frac{\partial\Delta(\lambda,c)}{\partial c} = -\lambda < 0 \text{ and } \lim_{c\to+\infty}\Delta(\lambda,c) = -\infty \text{ for } \lambda > 0,\\
&\left.\frac{\partial\Delta(\lambda,c)}{\partial\lambda}\right|_{\lambda=0} = -c < 0 \text{ for all } c > 0,\\
&\frac{\partial^2\Delta(\lambda,c)}{\partial\lambda^2} = d_2\int_{\mathbb{R}} J_2(y)y^2e^{-\lambda y}dy > 0 \text{ for all } \lambda \text{ and } c.
\end{aligned}$$

Thus, we have the following properties.

Lemma 1.1. *There exists a positive constant*

$$c^* = \inf_{\lambda>0}\left\{\frac{1}{\lambda}\left[d_2\left(\int_{\mathbb{R}} J_2(y)e^{-\lambda y}dy - 1\right) + s\right]\right\} \tag{6}$$

such that the following assertions hold.

(a) *If $0 < c < c^*$, then $\Delta(\lambda,c) > 0$ for $\lambda > 0$.*

(b) *If $c > c^*$, then $\Delta(\lambda,c) = 0$ has two positive real roots $\lambda_1 < \lambda_2$, and $\Delta(\cdot,c) < 0$ in (λ_1,λ_2) and $\Delta(\cdot,c) > 0$ in $(0,\lambda_1)\cup(\lambda_2,+\infty)$.*

(c) *If $c = c^*$, then there exists $\lambda^* > 0$ such that $\Delta(\lambda^*,c^*) = 0$ and $\Delta(\lambda,c^*) > 0$ for $\lambda \neq \lambda^*$.*

Now, our major result is as follows.

Theorem 1.1. *Assume that (J1)-(J2) hold. If $b \in (0,1)$ and*

$$m < \min\left\{ \frac{(1-b)^2(1+b)}{8b}, \frac{(1-b)^2(1+a)}{4b}, \frac{(1+a)^3}{8}\left(\sqrt{b^2+4\left(\frac{1-b}{1+a}\right)^2}-b\right)\right\}, \tag{7}$$

then (2) has traveling waves connecting $(1,0)$ and (u_2^*, u_2^*) for $c \geq c^*$. While (2) has no traveling waves connecting $(1,0)$ and (u_2^*, u_2^*) for $0 < c < c^*$.

Notably, our approach is also adaptable to the Holling–Tanner system with local diffusion and strong Allee effect, and results about spreading speed and the comparison principle are detailed in [28,29]. Regarding (1), the existence of traveling wave with the wave speed $c = c^*$ is proved via a limiting argument similar to Theorem 4.1 in [30], however, this procedure is omitted due to its complexity. In its place, we adopt an entirely different approach with [14], and offer a complete proof.

This paper is organized as follows. In Section 2, we present a theoretical framework for the existence of the positive solution of (4). Section 3 is dedicated to establishing the existence of a positive solution of (4)–(5) by constructing upper and lower solutions and a novel sequence for $c > c^*$. In Section 4, we rigorously validate the case $c = c^*$ by spreading speed theory and the comparison principle, and also deduce the nonexistence of the positive solution of (4)–(5), conclusively determining the minimal wave speed. In the final appendix section, we mention some crucial results used in this paper.

2. A general result

In this section, by Schauder's fixed point theorem and upper and lower solutions method, we reframe the quest to find a positive solution of (4) into an existence problem concerning a pair of upper and lower solutions. To begin, we define

$$X_b = \left\{(\phi,\psi) \in C\left(\mathbb{R},\mathbb{R}^2\right) : (1+b)/2 \leq \phi \leq 1 \text{ and } 0 \leq \psi \leq 1\right\},$$

and introduce upper and lower solutions.

Definition 2.1. Function pairs $\left(\overline{\phi},\overline{\psi}\right)$ and $\left(\underline{\phi},\underline{\psi}\right)$ in X_b are upper solution and lower solution of (4) if they satisfy

(a) $\underline{\phi}(\xi) \leq \overline{\phi}(\xi)$, $\underline{\psi}(\xi) \leq \overline{\psi}(\xi)$ for $\xi \in \mathbb{R}$;

(b) there exists a finite set $E = \left\{\xi_i : 1 \leq i \leq m\right\}$ such that for $\xi \in \mathbb{R}\backslash E$

$$d_1\mathcal{N}_1\left[\overline{\phi}\right](\xi) - c\overline{\phi}'(\xi) + \overline{\phi}(\xi)f\left(\overline{\phi},\underline{\psi}\right)(\xi) \leq 0, \tag{8}$$

$$d_1\mathcal{N}_1\left[\underline{\phi}\right](\xi) - c\underline{\phi}'(\xi) + \underline{\phi}(\xi)f\left(\underline{\phi},\overline{\psi}\right)(\xi) \geq 0, \tag{9}$$

$$d_2\mathcal{N}_2\left[\overline{\psi}\right](\xi) - c\overline{\psi}'(\xi) + \overline{\psi}(\xi)g\left(\overline{\phi},\overline{\psi}\right)(\xi) \leq 0, \tag{10}$$

$$d_2\mathcal{N}_2\left[\underline{\psi}\right](\xi) - c\underline{\psi}'(\xi) + \underline{\psi}(\xi)g\left(\underline{\phi},\underline{\psi}\right)(\xi) \geq 0. \tag{11}$$

Following this, we consider the nonlinear operators F_1 and F_2 on X_b defined by

$$F_1(\phi,\psi)(\xi) := \beta\phi(\xi) + d_1\mathcal{N}_1[\phi](\xi) + \phi(\xi)f(\phi,\psi)(\xi),$$
$$F_2(\phi,\psi)(\xi) := \beta\psi(\xi) + d_2\mathcal{N}_2[\psi](\xi) + \psi(\xi)g(\phi,\psi)(\xi),$$

where the positive constant β is large enough, ensuring that $F_1(\phi,\psi)$ increases in ϕ and decreases in ψ, while $F_2(\phi,\psi)$ increases in both ϕ and ψ for $(\phi,\psi)(\xi) \in X_b$, respectively. We also define the operators P_i, $i = 1,2$ as follows

$$P_i(\phi,\psi)(\xi) = \frac{1}{c}\int_{-\infty}^{\xi} e^{\frac{\beta(y-\xi)}{c}} F_i(\phi,\psi)(y)dy, \ \xi \in \mathbb{R}.$$

Set $P = (P_1,P_2)$, clearly, $P : X_b \to C\left(\mathbb{R},\mathbb{R}^2\right)$, and if $(\hat{\phi},\hat{\psi}) = P(\hat{\phi},\hat{\psi})$, then $(\hat{\phi},\hat{\psi})$ solves

$$\begin{cases} c\hat{\phi}'(\xi) = -\beta\hat{\phi}(\xi) + F_1(\hat{\phi},\hat{\psi})(\xi), \\ c\hat{\psi}'(\xi) = -\beta\hat{\psi}(\xi) + F_2(\hat{\phi},\hat{\psi})(\xi). \end{cases}$$

Hence the fixed point of P is a solution of (4). Therefore, it remains to show that P has a fixed point in X_b. To proceed, we select a constant $\mu \in (0,\beta/c)$ and define the space

$$B_\mu(\mathbb{R},\mathbb{R}^2) := \left\{ (\phi,\psi) \in X_b : |(\phi,\psi)|_\mu = \sup_{\xi\in\mathbb{R}}\left\{\max\left(|\phi(\xi)|,|\psi(\xi)|\right)\right\}e^{-\mu|\xi|} < +\infty \right\}.$$

Then $\left(B_\mu(\mathbb{R},\mathbb{R}^2), |\cdot|_\mu\right)$ is a Banach space from [31]. By applying Schauder's fixed point theorem, we will seek a positive solution of (4) in set

$$\Sigma = \left\{(\phi,\psi) \in X_b : \underline{\phi} \leq \phi \leq \overline{\phi},\ \underline{\psi} \leq \psi \leq \overline{\psi}\right\},$$

which is a non-empty convex, closed and bounded set in $(B_\mu(\mathbb{R},\mathbb{R}^2),|\cdot|_\mu)$.

Firstly, we show that $P(\Sigma)\subseteq\Sigma$. For $(\phi,\psi)\in\Sigma$, since (ϕ,ψ), $\left(\overline{\phi},\overline{\psi}\right)$ and $\left(\underline{\phi},\underline{\psi}\right)$ all belong to X_b, we have

$$F_1(\overline{\phi},\underline{\psi})(\xi)\geq F_1(\phi,\psi)(\xi)\geq F_1(\underline{\phi},\overline{\psi})(\xi)$$

by the choice of β. We assert that

$$\underline{\phi}(\xi)\leq P_1(\phi,\psi)(\xi)\leq\overline{\phi}(\xi).$$

Assume that $-\infty:=\xi_{m+1}<\xi_m<\xi_{m-1}<\cdots<\xi_1<\xi_0:=+\infty$ in set E of Definition 2.1, and that for $\xi_i\in E$

$$\underline{\phi}'(\xi_i^-)\leq\underline{\phi}'(\xi_i^+),\ \overline{\phi}'(\xi_i^+)\leq\overline{\phi}'(\xi_i^-),\ \underline{\psi}'(\xi_i^-)\leq\underline{\psi}'(\xi_i^+),\ \overline{\psi}'(\xi_i^+)\leq\overline{\psi}'(\xi_i^-).$$

Through integration by parts formula, we derive for $0\leq i\leq m$

$$\frac{1}{c}\int_{\xi_{i+1}}^{\xi_i}e^{\frac{\beta}{c}y}\left(c\phi'(y)+\beta\phi(y)\right)dy=e^{\frac{\beta}{c}\xi_i}\phi(\xi_i)-e^{\frac{\beta}{c}\xi_{i+1}}\phi(\xi_{i+1}).$$

For $\xi\in(\xi_{k+1},\xi_k)$ with $0\leq k\leq m$, from the definition of upper and lower solutions, we arrive at

$$\begin{aligned}P_1(\phi,\psi)(\xi)\geq P_1(\underline{\phi},\overline{\psi})(\xi)&=\frac{1}{c}\int_{-\infty}^{\xi}e^{\frac{\beta(y-\xi)}{c}}F_1(\underline{\phi},\overline{\psi})(y)dy\\&\geq\frac{1}{c}\int_{-\infty}^{\xi}e^{\frac{\beta(y-\xi)}{c}}\left(c\underline{\phi}'(y)+\beta\underline{\phi}(y)\right)dy\\&=\frac{1}{c}\left(\sum_{j=k+1}^{m}\int_{\xi_{j+1}}^{\xi_j}+\int_{\xi_{k+1}}^{\xi}\right)e^{\frac{\beta(y-\xi)}{c}}\left(c\underline{\phi}'(y)+\beta\underline{\phi}(y)\right)dy\\&\geq\underline{\phi}(\xi)-e^{\frac{\beta(\xi_{m+1}-\xi)}{c}}\underline{\phi}(\xi_{m+1})=\underline{\phi}(\xi)\end{aligned}$$

owing to $\xi_{m+1}=-\infty$. Similarly, we get

$$\begin{aligned}P_1(\phi,\psi)(\xi)\leq P_1(\overline{\phi},\underline{\psi})(\xi)&=\frac{1}{c}\int_{-\infty}^{\xi}e^{\frac{\beta(y-\xi)}{c}}F_1(\overline{\phi},\underline{\psi})(y)dy\\&\leq\frac{1}{c}\int_{-\infty}^{\xi}e^{\frac{\beta(y-\xi)}{c}}\left(c\overline{\phi}'(y)+\beta\overline{\phi}(y)\right)dy\\&=\frac{1}{c}\left(\sum_{j=k+1}^{m}\int_{\xi_{j+1}}^{\xi_j}+\int_{\xi_{k+1}}^{\xi}\right)e^{\frac{\beta(y-\xi)}{c}}\left(c\overline{\phi}'(y)+\beta\overline{\phi}(y)\right)dy\\&\leq\overline{\phi}(\xi)-e^{\frac{\beta(\xi_{m+1}-\xi)}{c}}\overline{\phi}(\xi_{m+1})=\overline{\phi}(\xi)\end{aligned}$$

owing to $\xi_{m+1}=-\infty$. Then in a similar way, we can verify that

$$\underline{\psi}(\xi)\leq P_2(\phi,\psi)(\xi)\leq\overline{\psi}(\xi).$$

Therefore, $P(\Sigma)\subseteq\Sigma$.

Next, we show that P is completely continuous with respect to the norm $|\cdot|_\mu$ by the choice of μ. First of all, we show the continuity of P on Σ with respect to the norm $|\cdot|_\mu$. Let $\Phi_1=(\phi_1,\psi_1)$ and $\Phi_2=(\phi_2,\psi_2)$ be in Σ, then a direct calculation yields that

$$\begin{aligned}&|F_1(\phi_1,\psi_1)(\xi)-F_1(\phi_2,\psi_2)(\xi)|\\=&|\beta\phi_1(\xi)+d_1\mathcal{N}_1[\phi_1](\xi)+\phi_1(\xi)f(\phi_1,\psi_1)(\xi)-\beta\phi_2(\xi)-d_1\mathcal{N}_1[\phi_2](\xi)-\phi_2(\xi)f(\phi_2,\psi_2)(\xi)|\\\leq&\left(\beta+d_1+C\right)|\phi_1(\xi)-\phi_2(\xi)|+C|\psi_1(\xi)-\psi_2(\xi)|+d_1\int_{\mathbb{R}}J_1(\xi-\tau)|\phi_1(\tau)-\phi_2(\tau)|d\tau,\end{aligned}$$

where

$$C=\sup_{(\phi,\psi)\in X_b}\left\{f(\phi,\psi)+\phi\frac{\partial f(\phi,\psi)}{\partial\phi},\ \phi\frac{\partial f(\phi,\psi)}{\partial\psi}\right\}.$$

Furthermore, we have

$$\begin{aligned}&|P_1(\phi_1,\psi_1)(\xi)-P_1(\phi_2,\psi_2)(\xi)|e^{-\mu|\xi|}\\=&\frac{1}{c}\left|\int_{-\infty}^{\xi}e^{\frac{\beta(y-\xi)}{c}}\left(F_1(\phi_1,\psi_1)(y)-F_1(\phi_2,\psi_2)(y)\right)dy\right|e^{-\mu|\xi|}\\\leq&\frac{e^{-\mu|\xi|}}{c}\int_{-\infty}^{\xi}e^{\frac{\beta(y-\xi)}{c}}|F_1(\phi_1,\psi_1)(y)-F_1(\phi_2,\psi_2)(y)|dy\\\leq&\frac{\left(\beta+d_1+C\right)e^{-\mu|\xi|}}{c}\int_{-\infty}^{\xi}e^{\frac{\beta(y-\xi)}{c}}|\phi_1(y)-\phi_2(y)|dy\\&+\frac{Ce^{-\mu|\xi|}}{c}\int_{-\infty}^{\xi}e^{\frac{\beta(y-\xi)}{c}}|\psi_1(y)-\psi_2(y)|dy\end{aligned}$$

$$+\frac{d_1 e^{-\mu|\xi|}}{c}\int_{-\infty}^{\xi} e^{\frac{\beta(y-\xi)}{c}}\left(\int_{\mathbb{R}} J_1(y-\tau)|\phi_1(\tau)-\phi_2(\tau)|d\tau\right)dy$$
$$=I_1+I_2+I_3.$$

Now, let us continue estimating I_1, I_2 and I_3, respectively. Note that $|y|-|\xi|\le \xi-y$ for all $y\le\xi$, we have

$$\int_{-\infty}^{\xi} e^{\frac{\beta(y-\xi)}{c}} e^{\mu|y|}e^{-\mu|\xi|}dy \le \int_{-\infty}^{\xi} e^{\frac{(\beta-c\mu)(y-\xi)}{c}}dy=\frac{c}{\beta-c\mu}. \tag{12}$$

For I_1, we utilize (12) to get

$$\begin{aligned}
I_1 &= \frac{\left(\beta+d_1+C\right)e^{-\mu|\xi|}}{c}\int_{-\infty}^{\xi} e^{\frac{\beta(y-\xi)}{c}}|\phi_1(y)-\phi_2(y)|dy\\
&= \frac{\left(\beta+d_1+C\right)e^{-\mu|\xi|}}{c}\int_{-\infty}^{\xi} e^{\frac{\beta(y-\xi)}{c}}e^{\mu|y|}|\phi_1(y)-\phi_2(y)|e^{-\mu|y|}dy\\
&\le \frac{|\Phi_1-\Phi_2|_\mu\left(\beta+d_1+C\right)}{c}\int_{-\infty}^{\xi} e^{\frac{\beta(y-\xi)}{c}}e^{\mu|y|}e^{-\mu|\xi|}dy\\
&\le \frac{\beta+d_1+C}{\beta-c\mu}|\Phi_1-\Phi_2|_\mu.
\end{aligned}$$

For I_2, similarly, we get for $\mu\in(0,\beta/c)$

$$I_2\le\frac{C}{\beta-c\mu}|\Phi_1-\Phi_2|_\mu.$$

For I_3, we also have

$$\begin{aligned}
I_3 &= \frac{d_1 e^{-\mu|\xi|}}{c}\int_{-\infty}^{\xi} e^{\frac{\beta(y-\xi)}{c}}\left(\int_{\mathbb{R}} J_1(y-\tau)|\phi_1(\tau)-\phi_2(\tau)|d\tau\right)dy\\
&\le \frac{d_1 e^{-\mu|\xi|}|\Phi_1-\Phi_2|_\mu}{c}\int_{-\infty}^{\xi} e^{\frac{\beta(y-\xi)}{c}}\left(\int_{\mathbb{R}} J_1(z)e^{\mu|y-z|}dz\right)dy\\
&\le \frac{d_1|\Phi_1-\Phi_2|_\mu}{c}\left(\int_{\mathbb{R}} J_1(z)e^{\mu|z|}dz\right)\int_{-\infty}^{\xi} e^{\frac{\beta(y-\xi)}{c}}e^{\mu|y|}e^{-\mu|\xi|}dy\\
&\le \frac{d_1}{\beta-c\mu}\left(\int_{\mathbb{R}} J_1(z)e^{\mu|z|}dz\right)|\Phi_1-\Phi_2|_\mu<\infty
\end{aligned}$$

owing to *(J2)*. Based on the preceding three estimates, we infer that

$$|P_1(\phi_1,\psi_1)(\xi)-P_1(\phi_2,\psi_2)(\xi)|e^{-\mu|\xi|}\le M_1|\Phi_1-\Phi_2|_\mu,$$

where

$$M_1=\frac{\beta+d_1+2C+d_1\int_{\mathbb{R}} J_1(z)e^{\mu|z|}dz}{\beta-c\mu}.$$

Similarly, there exists a positive constant M_2 such that

$$|P_2(\phi_1,\psi_1)(\xi)-P_2(\phi_2,\psi_2)(\xi)|e^{-\mu|\xi|}\le M_2|\Phi_1-\Phi_2|_\mu.$$

Hence P is continuous on Σ with respect to the norm $|\cdot|_\mu$. Simultaneously, for $(\phi,\psi)\in\Sigma$,

$$\begin{aligned}
\left|\frac{d}{d\xi}P_1(\phi,\psi)(\xi)\right| &= \left|-\frac{\beta}{c}P_1(\phi,\psi)(\xi)+\frac{1}{c}F_1(\phi,\psi)(\xi)\right|\le\frac{\beta}{c}\overline{\phi}(\xi)+\frac{1}{c}F_1(1,0)\le\frac{2\beta}{c},\\
\left|\frac{d}{d\xi}P_2(\phi,\psi)(\xi)\right| &= \left|-\frac{\beta}{c}P_2(\phi,\psi)(\xi)+\frac{1}{c}F_2(\phi,\psi)(\xi)\right|\le\frac{\beta}{c}\overline{\psi}(\xi)+\frac{1}{c}F_2(1,1)\le\frac{2\beta}{c}.
\end{aligned}$$

Thus, $P(\Sigma)$ is equicontinuous.

Using an argument in Lemma 3.4 [32], one can assert that $P:\Sigma\to\Sigma$ is compact. Actually, for any $(\phi,\psi)\in\Sigma$ and $n\in\mathbb{N}$, we define

$$P^n(\phi,\psi)(\xi)=\begin{cases} P(\phi,\psi)(-n), & \xi<-n,\\ P(\phi,\psi)(\xi), & \xi\in[-n,n],\\ P(\phi,\psi)(n), & \xi>n.\end{cases}$$

It is clear that $P^n:\Sigma\to B_\mu(\mathbb{R},\mathbb{R}^2)$ is continuous, and $P^n(\Sigma)$ is equicontinuous and uniformly bounded with respect to norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R},\mathbb{R}^2)$, implying that P^n is compact operator. Furthermore, owing to $P_1(\phi,\psi)\le\overline{\phi}\le 1$, we have

$$|P_1(\phi,\psi)(\xi)-P_1^n(\phi,\psi)(\xi)|_\mu\le\sup_{\xi\in(-\infty,-n)\cup(n,+\infty)}|P_1(\phi,\psi)(\xi)-P_1^n(\phi,\psi)(\xi)|e^{-\mu n}\le 2e^{-\mu n},$$

which gives

$$|P_1(\phi,\psi)(\xi)-P_1^n(\phi,\psi)(\xi)|_\mu \to 0 \text{ as } n\to+\infty.$$

Similarly, we can prove that

$$|P_2(\phi,\psi)(\xi)-P_2^n(\phi,\psi)(\xi)|_\mu \to 0 \text{ as } n\to+\infty.$$

Thus, $|P(\phi,\psi)(\xi)-P^n(\phi,\psi)(\xi)|_\mu \to 0$ as $n\to+\infty$. From Proposition 2.12 in [33], P is compact operator with respect to the norm $|\cdot|_\mu$. Now, by Schauder's fixed point theorem, there exists $(\hat{\phi},\hat{\psi})\in\Sigma$ such that $P(\hat{\phi},\hat{\psi})=(\hat{\phi},\hat{\psi})$, which implies that $(\hat{\phi},\hat{\psi})$ is a fixed point of P in Σ. Therefore, we conclude the following lemma.

Lemma 2.1. *If (4) has a pair of upper and lower solutions $\left(\overline{\phi},\overline{\psi}\right)$ and $\left(\underline{\phi},\underline{\psi}\right)$ satisfying*

$$\underline{\phi}'(\xi_i^-)\le\underline{\phi}'(\xi_i^+),\ \overline{\phi}'(\xi_i^+)\le\overline{\phi}'(\xi_i^-),\ \underline{\psi}'(\xi_i^-)\le\underline{\psi}'(\xi_i^+),\ \overline{\psi}'(\xi_i^+)\le\overline{\psi}'(\xi_i^-)\ \text{for } \xi_i\in E.$$

Then it has a solution (ϕ,ψ) satisfying for $\xi\in\mathbb{R}$

$$\underline{\phi}(\xi)\le\phi(\xi)\le\overline{\phi}(\xi),\quad \underline{\psi}(\xi)\le\psi(\xi)\le\overline{\psi}(\xi).$$

3. The existence of traveling waves for $c>c^*$

The primary aim of this section is to demonstrate the existence of traveling waves with $c>c^*$ of (2), where c^* is defined in (6). Firstly, a pair of appropriate upper and lower solutions is constructed, following which we immediately obtain a positive solution $(\phi,\psi)(\xi)$ of (4). Subsequently, using a novel sequence we investigate the asymptotic behavior of this positive solution at $\xi=+\infty$ via the squeeze method [10,34], confirming the existence of traveling waves for $c>c^*$.

3.1. Upper and lower solutions

To identify appropriate upper and lower solutions, we define

$$\Pi(\lambda,c)=d_1\left(\int_{\mathbb{R}}J_1(y)\mathrm{e}^{-\lambda y}\mathrm{d}y-1\right)-c\lambda,$$

then one can easily check that for $c>c^*$

$$\Pi(0,c)=0 \text{ and } \left.\frac{\partial\Pi(\lambda,c)}{\partial\lambda}\right|_{\lambda=0}=-c<0.$$

Thus, we choose $\eta\in(0,\lambda_1)$ to satisfy $\Pi(\eta,c)<0$. We now introduce functions

$$\overline{\phi}(\xi)=1,\qquad \underline{\phi}(\xi)=\begin{cases}\dfrac{1+b}{2}, & \xi>0,\\ 1-\dfrac{1-b}{2}e^{\eta\xi}, & \xi\le 0,\end{cases}$$

$$\overline{\psi}(\xi)=\begin{cases}1, & \xi>0,\\ e^{\lambda_1\xi}, & \xi\le0,\end{cases}\qquad \underline{\psi}(\xi)=\begin{cases}0, & \xi>\xi_1,\\ e^{\lambda_1\xi}\left(1-re^{\varepsilon\xi}\right), & \xi\le\xi_1,\end{cases}$$

where $\xi_1=-(1/\varepsilon)\ln r$, and ε, r satisfy

$$0<\varepsilon<\min\left\{\lambda_1,\ \lambda_2-\lambda_1\right\},\quad r>\max\left\{1,\ \frac{-2s}{(1+b)\Delta(\lambda_1+\varepsilon,c)}\right\}.$$

Lemma 3.1. *Suppose (7) holds. For $c>c^*$, $(\overline{\phi},\overline{\psi})$ and $(\underline{\phi},\underline{\psi})$ satisfy (8) and (9).*

Proof. Recalling that

$$f(\phi,\psi)=(1-\phi)\left(\frac{\phi}{b}-1\right)-\frac{m\psi}{\phi+a\psi},$$

since $\overline{\phi}(\xi)=1$ and $\underline{\psi}(\xi)\ge0$ for $\xi\in\mathbb{R}$, then

$$d_1\mathcal{N}_1\left[\overline{\phi}\right](\xi)-c\overline{\phi}'(\xi)+\overline{\phi}(\xi)f\left(\overline{\phi},\underline{\psi}\right)(\xi)=-\frac{m\underline{\psi}(\xi)}{1+a\underline{\psi}(\xi)}\le0,$$

which implies (8) holds. For (9), since $\underline{\phi}(\xi)$ is non-increasing in $\mathbb{R}$, then

$$\int_{\mathbb{R}}J_1(\xi-y)\underline{\phi}(y)dy\ge\frac{1+b}{2}\int_{\mathbb{R}}J_1(\xi-y)dy=\frac{1+b}{2},$$

and

$$\int_{\mathbb{R}} J_1(\xi-y)\underline{\phi}(y)dy \geq \int_{\mathbb{R}} J_1(\xi-y)\left(1-\frac{1-b}{2}e^{\eta y}\right)dy = 1-\frac{1-b}{2}e^{\eta\xi}\int_{\mathbb{R}} J_1(y)e^{-\eta y}dy.$$

Hence we have

$$\int_{\mathbb{R}} J_1(\xi-y)\underline{\phi}(y)dy \geq \max\left\{\frac{1+b}{2},\ 1-\frac{1-b}{2}e^{\eta\xi}\int_{\mathbb{R}} J_1(y)e^{-\eta y}dy\right\}. \tag{13}$$

If $\xi>0$, then $\underline{\phi}(\xi)=(1+b)/2$ and $\overline{\psi}(\xi)=1$. It follows from (13) that

$$\mathcal{N}_1\left[\underline{\phi}\right](\xi)=\int_{\mathbb{R}} J_i(\xi-y)\underline{\phi}(y)dy-\underline{\phi}(\xi)\geq 0.$$

By (7), we further get

$$d_1\mathcal{N}_1\left[\underline{\phi}\right](\xi)-c\underline{\phi}'(\xi)+\underline{\phi}(\xi)f\left(\underline{\phi},\overline{\psi}\right)(\xi)\geq\underline{\phi}\left(\frac{(1-b)^2}{4b}-\frac{2m}{1+b+2a}\right)>0.$$

If $\xi<0$, then $\underline{\phi}=1-(1-b)e^{\eta\xi}/2$ and $\overline{\psi}=e^{\lambda_1\xi}$. It follows from (13) that

$$\begin{aligned}\mathcal{N}_1\left[\underline{\phi}\right](\xi)=&\int_{\mathbb{R}} J_i(\xi-y)\underline{\phi}(y)dy-\underline{\phi}(\xi)\\ \geq&1-\frac{1-b}{2}e^{\eta\xi}\int_{\mathbb{R}} J_1(y)e^{-\eta y}dy-\left(1-\frac{1-b}{2}e^{\eta\xi}\right)\\ =&\frac{1-b}{2}e^{\eta\xi}\left(1-\int_{\mathbb{R}} J_1(y)e^{-\eta y}dy\right).\end{aligned}$$

Thus, we have

$$\begin{aligned}&d_1\mathcal{N}_1\left[\underline{\phi}\right](\xi)-c\underline{\phi}'(\xi)+\underline{\phi}(\xi)f\left(\underline{\phi},\overline{\psi}\right)(\xi)\\ \geq&\frac{1-b}{2}e^{\eta\xi}\left[d_1\left(1-\int_{\mathbb{R}} J_1(y)e^{-\eta y}dy\right)+c\eta\right]\\ &+\underline{\phi}e^{\eta\xi}\left[\frac{1-b}{2}\left(\frac{1}{b}-\frac{1-b}{2b}e^{\eta\xi}-1\right)-\frac{me^{(\lambda_1-\eta)\xi}}{\underline{\phi}(\xi)+ae^{\lambda_1\xi}}\right]\\ =&-\frac{1-b}{2}e^{\eta\xi}\Pi(\eta,c)+\underline{\phi}e^{\eta\xi}K_1\end{aligned}$$

where

$$K_1=\frac{1-b}{2}\left(\frac{1}{b}-\frac{1-b}{2b}e^{\eta\xi}-1\right)-\frac{me^{(\lambda_1-\eta)\xi}}{\underline{\phi}(\xi)+ae^{\lambda_1\xi}}.$$

Since $0<\eta<\lambda_1$, then $(\lambda_1-\eta)\xi<0$. Owing to $\underline{\phi}\geq(1+b)/2$ over $\mathbb{R}$ and (7), we get

$$K_1\geq\frac{(1-b)^2}{4b}-\frac{2m}{1+b}>0.$$

On the other hand, $\Pi(\eta,c)<0$ by the choice of η, which ensures that (9) holds. Therefore, we complete the proof. □

Lemma 3.2. *For $c>c^*$, $(\overline{\phi},\overline{\psi})$ and $(\underline{\phi},\underline{\psi})$ satisfy (10) and (11).*

Proof. It is easy to check that

$$\int_{\mathbb{R}} J_1(\xi-y)\overline{\psi}(y)dy\leq\min\left\{1,e^{\lambda_1\xi}\int_{\mathbb{R}} J_2(y)e^{-\lambda_1 y}dy\right\}.$$

If $\xi>0$, then $\overline{\psi}(\xi)=\overline{\phi}(\xi)=1$. Recall that $g(\phi,\psi)=s(1-\psi/\phi)$, clearly, we have

$$d_2\mathcal{N}_2\left[\overline{\psi}\right](\xi)-c\overline{\psi}'(\xi)+\overline{\psi}(\xi)g\left(\overline{\phi},\overline{\psi}\right)(\xi)\leq 0.$$

If $\xi<0$, then $\overline{\psi}(\xi)=e^{\lambda_1\xi}$ and $\overline{\phi}(\xi)=1$. Note that $\Delta(\lambda_1,c)=0$ for $c>c^*$, we arrive at

$$\begin{aligned}&d_2\mathcal{N}_2\left[\overline{\psi}\right](\xi)-c\overline{\psi}'(\xi)+\overline{\psi}(\xi)g\left(\overline{\phi},\overline{\psi}\right)(\xi)\\ \leq&e^{\lambda_1\xi}\left[d_2\left(\int_{\mathbb{R}} J_2(y)e^{-\lambda_1 y}dy-1\right)-c\lambda_1+s\right]-se^{2\lambda_1\xi}\\ =&e^{\lambda_1\xi}\Delta(\lambda_1,c)-se^{2\lambda_1\xi}=-se^{2\lambda_1\xi}\leq 0.\end{aligned}$$

Hence (10) holds. For (11), it is easy to check that

$$\int_{\mathbb{R}} J_1(\xi-y)\underline{\psi}(y)dy\geq\max\left\{0,\int_{\mathbb{R}} J_2(y)e^{\lambda_1(\xi-y)}\left(1-re^{\varepsilon(\xi-y)}\right)dy\right\}.$$

If $\xi > \xi_1$, $\underline{\psi}(\xi) = 0$, clearly, (11) holds. If $\xi < \xi_1 < 0$, then

$$\underline{\psi}(\xi) = e^{\lambda_1 \xi}\left(1 - re^{\varepsilon\xi}\right), \quad \underline{\psi}'(\xi) = e^{\lambda_1 \xi}\left(\lambda_1 - r(\lambda_1+\varepsilon)e^{\varepsilon\xi}\right).$$

Since $\varepsilon < \lambda_2 - \lambda_1$, we have $\Delta(\lambda_1+\varepsilon, c) < 0$ for $c > c^*$. Owing to $\underline{\phi} \geq (1+b)/2$ over $\mathbb{R}$ and the choice of r, we arrive at

$$\begin{aligned}
& d_2\mathcal{N}_2\left[\underline{\psi}\right](\xi) - c\underline{\psi}'(\xi) + \underline{\psi}(\xi) g\left(\underline{\phi}, \underline{\psi}\right)(\xi) \\
\geq & d_2\left(\int_{\mathbb{R}} J_2(y) e^{\lambda_1(\xi-y)}\left(1 - re^{\varepsilon(\xi-y)}\right)dy - e^{\lambda_1\xi}\left(1 - re^{\varepsilon\xi}\right)\right) \\
& - ce^{\lambda_1\xi}\left(\lambda_1 - r(\lambda_1+\varepsilon)e^{\varepsilon\xi}\right) + se^{\lambda_1\xi}\left(1 - re^{\varepsilon\xi}\right) - \frac{se^{2\lambda_1\xi}\left(1 - re^{\varepsilon\xi}\right)^2}{\underline{\phi}(\xi)} \\
\geq & e^{\lambda_1\xi}\left[d_2\left(\int_{\mathbb{R}} J_2(y)e^{-\lambda_1 y}dy - 1\right) - c\lambda_1 + s\right] - \frac{2se^{2\lambda_1\xi}}{1+b} \\
& - re^{(\lambda_1+\varepsilon)\xi}\left[d_2\left(\int_{\mathbb{R}} J_2(y)e^{-(\lambda_1+\varepsilon)y}dy - 1\right) - c(\lambda_1+\varepsilon) + s\right] \\
= & e^{\lambda_1\xi}\Delta(\lambda_1, c) - re^{(\lambda_1+\varepsilon)\xi}\Delta(\lambda_1+\varepsilon, c) - \frac{2se^{2\lambda_1\xi}}{1+b} \\
= & e^{(\lambda_1+\varepsilon)\xi}\left(-r\Delta(\lambda_1+\varepsilon, c) - \frac{2se^{(\lambda_1-\varepsilon)\xi}}{1+b}\right) \\
\geq & e^{(\lambda_1+\varepsilon)\xi}\left(-r\Delta(\lambda_1+\varepsilon, c) - \frac{2s}{1+b}\right) > 0.
\end{aligned}$$

Hence (11) holds. Therefore, we complete the proof. □

3.2. Asymptotic behavior

In the light of upper and lower solutions, a positive solution $(\phi, \psi)(\xi)$ of (4) is derived with the aid of Lemma 2.1. Our primary objective in this subsection is to investigate the asymptotic behavior of the positive solution at $\xi = +\infty$.

We firstly give the existence result on the positive solution of (4).

Theorem 3.1. *Suppose (7) holds, (4) has a positive solution $(\phi, \psi)(\xi)$ satisfying $(1+b)/2 < \phi(\xi) < 1$ and $0 < \psi(\xi) < 1$ over $\mathbb{R}$ for all $c > c^*$.*

Proof. From Lemma 2.1, (4) has a positive solution $(\phi, \psi)(\xi)$ satisfying $\underline{\phi}(\xi) \leq \phi(\xi) \leq \overline{\phi}(\xi)$ and $\underline{\psi}(\xi) \leq \psi(\xi) \leq \overline{\psi}(\xi)$ over $\mathbb{R}$. We firstly show that $\phi(\xi) > (1+b)/2$ over $\mathbb{R}$. For contradiction, assume that there exists a $\xi_0 \in \mathbb{R}$ such that $\phi(\xi_0) = (1+b)/2$. Then $\phi'(\xi_0) = 0$ owing to $\underline{\phi}(\xi) \geq (1+b)/2$ over $\mathbb{R}$. By ϕ-equation of (4), due to $\overline{\psi}(\xi) \leq 1$ and (7), we get

$$0 \geq -d_1\mathcal{N}_1[\phi](\xi_0) = \phi(\xi_0) f(\phi, \psi)(\xi_0) \geq \phi(\xi_0) f((1+b)/2, 1) > 0.$$

Thus, $\phi(\xi) > (1+b)/2$ over $\mathbb{R}$. We also verify that $\psi(\xi) > 0$ over $\mathbb{R}$ by contradiction. Assume that there exists a $\xi_0 \in \mathbb{R}$ such that $\psi(\xi_0) = 0$, then $\psi'(\xi_0) = 0$ owing to $\underline{\psi}(\xi)$ being non-negative over $\mathbb{R}$. By ψ-equation of (4), we get

$$\int_{\mathbb{R}} J_2(\xi_0 - y)\psi(y)dy = 0 \text{ for all } y \in \mathbb{R}.$$

Thus, $\psi(\xi) \equiv 0$ over $\mathbb{R}$, which contradicts to $\psi(\xi) > \underline{\psi}(\xi) > 0$ for $\xi < \xi_1$. We further show that $\phi(\xi) < 1$ over $\mathbb{R}$. Contrarily, if there exists a $\xi_0 \in \mathbb{R}$ such that $\phi(\xi_0) = 1$, then $\phi'(\xi_0) = 0$ owing to $\phi(\xi) \leq 1$ over $\mathbb{R}$. By ϕ-equation of (4), we get

$$0 \leq -d_1\mathcal{N}_1[\phi](\xi_0) = -\frac{m\psi(\xi_0)}{1 + a\psi(\xi_0)} < 0,$$

implying that $\phi(\xi) < 1$ over $\mathbb{R}$. The instance where $\psi(\xi) < 1$ can be treated in a similar way. Therefore, we conclude the proof. □

Regarding the positive solution $(\phi, \psi)(\xi)$ obtained in the preceding theorem, we have

$$\begin{aligned}
1 &= \lim_{\xi\to-\infty} \underline{\phi}(\xi) \leq \liminf_{\xi\to-\infty} \phi(\xi) \leq \limsup_{\xi\to-\infty} \phi(\xi) \leq \lim_{\xi\to-\infty} \overline{\phi}(\xi) = 1, \\
0 &= \lim_{\xi\to-\infty} \underline{\psi}(\xi) \leq \liminf_{\xi\to-\infty} \psi(\xi) \leq \limsup_{\xi\to-\infty} \psi(\xi) \leq \lim_{\xi\to-\infty} \overline{\psi}(\xi) = 0,
\end{aligned}$$

which gives $(\phi, \psi)(-\infty) = (1, 0)$. Next, our main goal is to prove

$$\lim_{\xi\to+\infty} \phi(\xi) = \lim_{\xi\to+\infty} \psi(\xi) = u_2^*.$$

To do it, we define

$$\begin{aligned}
\phi_+ &= \limsup_{\xi\to+\infty} \phi(\xi), \quad \phi_- = \liminf_{\xi\to+\infty} \phi(\xi), \\
\psi_+ &= \limsup_{\xi\to+\infty} \psi(\xi), \quad \psi_- = \liminf_{\xi\to+\infty} \psi(\xi).
\end{aligned}$$

Lemma 3.3. $\phi_- \le \psi_- \le \psi_+ \le \phi_+$.

Proof. We first show that $\phi_- \le \psi_-$. For contradiction, we assume that $\phi_- > \psi_-$.

When $\psi(\xi)$ exhibits eventually monotone, then $\psi(+\infty)$ exists and

$$\int_0^{+\infty} \psi'(s)ds = \psi(+\infty) - \psi(0) < \infty$$

since $\psi(\xi)$ is bounded on $\mathbb{R}$. Note that if $\psi'(\xi) \le 0$ for $\xi \gg 1$, then $\limsup_{\xi\to+\infty} \psi'(\xi) = 0$, while if $\psi'(\xi) \ge 0$ for $\xi \gg 1$, then $\liminf_{\xi\to+\infty} \psi'(\xi) = 0$. Thus, we always can find a sequence $\{\xi_n\}_{n=0}^{+\infty}$, with $\xi_n \to +\infty$ as $n \to +\infty$, such that

$$\lim_{n\to+\infty} \psi(\xi_n) = \psi_- = \psi_+ < \phi_- \text{ and } \lim_{n\to+\infty} \psi'(\xi_n) = 0. \tag{14}$$

Integrating ψ-equation of (4) from 0 to ξ_n, we obtain

$$c[\psi(\xi_n) - \psi(0)] - d_2 \int_0^{\xi_n} \mathcal{N}_2[\psi](\xi)d\xi = \int_0^{\xi_n} \psi(\xi)g(\phi,\psi)(\xi)d\xi. \tag{15}$$

Direct computation yields

$$\begin{aligned}
\int_0^{\xi_n} \mathcal{N}_2[\psi](\xi)d\xi &= \int_0^{\xi_n}\int_{\mathbb{R}} J_2(y)[\psi(\xi-y)-\psi(\xi)]dyd\xi \\
&= \int_0^{\xi_n}\int_{\mathbb{R}} J_2(y)(-y)\int_0^1 \psi'(\xi-\tau y)d\tau dyd\xi \\
&= \int_{\mathbb{R}} J_2(y)(-y)\int_0^1\int_0^{\xi_n} \psi'(\xi-\tau y)d\xi d\tau dy \\
&= \int_{\mathbb{R}} J_2(y)(-y)\int_0^1 [\psi(\xi_n-\tau y)-\psi(-\tau y)]d\tau dy.
\end{aligned}$$

Since $0 < \psi(\xi) < 1$ over $\mathbb{R}$, from *(J1)* and *(J2)*, we further arrive at

$$\begin{aligned}
&\int_{\mathbb{R}} J_2(y)(-y)\int_0^1 [\psi(\xi_n-\tau y)-\psi(-\tau y)]d\tau dy \\
\le& 2\int_{\mathbb{R}} J_2(y)|y|dy \le 4\int_{\mathbb{R}} J_2(y)e^y dy < \infty.
\end{aligned}$$

On the other hand, we have by (14)

$$\liminf_{n\to+\infty} g(\phi,\psi)(\xi_n) \ge g(\phi_-,\psi_-) = s\left(1-\frac{\psi_-}{\phi_-}\right) > 0. \tag{16}$$

Hence the left-hand side of (15) is bounded, whereas the right-hand side of (15) is unbounded, which leads to a contradiction.

Next, we examine another case wherein $\psi(\xi)$ is oscillatory as $\xi \to +\infty$. We can then find a sequence $\{\xi_n\}_{n=0}^{+\infty}$ of the minimal points of $\psi(\xi)$, with $\xi_n \to +\infty$ as $n \to +\infty$, such that

$$\lim_{n\to+\infty} \psi(\xi_n) = \psi_- < \phi_- \text{ and } \psi'(\xi_n) = 0. \tag{17}$$

We also have (16) and get what we desired.

Finally, we proceed by contradiction once again to prove the inequality $\psi_+ \le \phi_+$. Assume that $\psi_+ > \phi_+$, similar to (14) and (17), one always can choose sequence $\{\xi_n\}_{n=0}^{+\infty}$, with $\xi_n \to +\infty$ as $n \to +\infty$, such that

$$\lim_{n\to+\infty} \psi(\xi_n) = \psi_+ > \phi_+ \text{ and } \lim_{n\to+\infty} \psi'(\xi_n) = 0.$$

At this moment, (15) remains valid, whereas

$$\limsup_{n\to+\infty} g(\phi,\psi)(\xi_n) \le g(\phi_+,\psi_+) = s\left(1-\frac{\psi_+}{\phi_+}\right) < 0.$$

Hence whether $\psi(\xi)$ is eventually monotone or oscillatory as $\xi \to +\infty$, a contradiction arises. Therefore, we complete the proof. □

Lemma 3.4. *Suppose (7) holds, we have* $(1+b)/2 < \phi_- \le \phi_+ < 1$.

Proof. We omit this proof here, as it follows directly from an argument analogous to Lemma 3.3. □

We are in this position to state the existence theorem for $c > c^*$.

Theorem 3.2. *Under (7) and* $b \in (0,1)$, *(2) has traveling waves connecting* $(1,0)$ *and* (u_2^*, u_2^*) *for* $c > c^*$.

Proof. We define a sequence $\{\gamma_n\}_{n=-1}^{+\infty}$, where

$$\begin{cases} \gamma_{-1} = \dfrac{1+b}{2}, \quad \gamma_0 = 1, \\ \gamma_{n+1} = \dfrac{1}{2}\left(b + 1 + \sqrt{(1-b)^2 - \dfrac{4mb\gamma_n}{\gamma_{n-1} + a\gamma_n}}\right). \end{cases}$$

Under (7) and $b \in (0,1)$, γ_n is well-defined and $(1+b)/2 \le \gamma_n \le 1$ for all n. Moreover, note that

$$(1-\gamma_{n+1})\left(\frac{\gamma_{n+1}}{b} - 1\right) = \frac{m\gamma_n}{\gamma_{n-1} + a\gamma_n},$$

and $(1-\gamma)(\gamma/b - 1)$ decreases with respect to γ in $\left[(1+b)/2, 1\right]$, we first show that the following claims hold.

Claim 1. *Under (7), the sequences $\{\gamma_{2n}\}_{n=0}^{+\infty}$ and $\{\gamma_{2n-1}\}_{n=0}^{+\infty}$ are adjacent, that is*

$$\gamma_{-1} < \gamma_1 < \cdots < \gamma_{2n-1} < \cdots < u_2^* < \cdots < \gamma_{2n} < \cdots < \gamma_2 < \gamma_0. \tag{18}$$

Note that $\gamma_{-1} < \gamma_1 < u_2^* < \gamma_2 < \gamma_0$ holds since

$$(1-\gamma_1)\left(\frac{\gamma_1}{b} - 1\right) = \frac{2m}{1+b+2a} > \frac{m}{1+a} = (1-u_2^*)\left(\frac{u_2^*}{b} - 1\right),$$

$$(1-\gamma_1)\left(\frac{\gamma_1}{b} - 1\right) = \frac{2m}{1+b+2a} < \frac{(1-b)^2}{4b} = (1-\gamma_{-1})\left(\frac{\gamma_{-1}}{b} - 1\right),$$

$$(1-\gamma_2)\left(\frac{\gamma_2}{b} - 1\right) = \frac{m\gamma_1}{\gamma_0 + a\gamma_1} < \frac{m}{1+a} = (1-u_2^*)\left(\frac{u_2^*}{b} - 1\right),$$

$$(1-\gamma_2)\left(\frac{\gamma_2}{b} - 1\right) = \frac{m\gamma_1}{\gamma_0 + a\gamma_1} > 0 = (1-\gamma_0)\left(\frac{\gamma_0}{b} - 1\right).$$

Suppose

$$\gamma_{n-1} < \gamma_{n+1} < u_2^* < \gamma_{n+2} < \gamma_n \text{ with } n = 2k,\ k \in \mathbb{N}, \tag{19}$$

we aim to prove

$$\gamma_{n+1} < \gamma_{n+3} < u_2^* < \gamma_{n+4} < \gamma_{n+2} \text{ with } n = 2k,\ k \in \mathbb{N}. \tag{20}$$

Due to $\gamma_{n+2} > u_2^* > \gamma_{n+1}$, we have

$$(1-\gamma_{n+3})\left(\frac{\gamma_{n+3}}{b} - 1\right) = \frac{m\gamma_{n+2}}{\gamma_{n+1} + a\gamma_{n+2}} > \frac{m}{1+a} = (1-u_2^*)\left(\frac{u_2^*}{b} - 1\right),$$

which implies $u_2^* > \gamma_{n+3}$. Thanks to $\gamma_{n+2} > u_2^* > \gamma_{n+3}$, we further get

$$(1-\gamma_{n+4})\left(\frac{\gamma_{n+4}}{b} - 1\right) = \frac{m\gamma_{n+3}}{\gamma_{n+2} + a\gamma_{n+3}} < \frac{m}{1+a} = (1-u_2^*)\left(\frac{u_2^*}{b} - 1\right),$$

implying that $\gamma_{n+4} > u_2^*$. On the other hand, from (19), we have

$$(1-\gamma_{n+3})\left(\frac{\gamma_{n+3}}{b} - 1\right) = \frac{m\gamma_{n+2}}{\gamma_{n+1} + a\gamma_{n+2}} < \frac{m\gamma_n}{\gamma_{n-1} + a\gamma_n} = (1-\gamma_{n+1})\left(\frac{\gamma_{n+1}}{b} - 1\right),$$

hence $\gamma_{n+3} > \gamma_{n+1}$. Due to $\gamma_{n+3} > \gamma_{n+1}$ and $\gamma_n > \gamma_{n+2}$, we further get

$$(1-\gamma_{n+4})\left(\frac{\gamma_{n+4}}{b} - 1\right) = \frac{m\gamma_{n+3}}{\gamma_{n+2} + a\gamma_{n+3}} > \frac{m\gamma_{n+1}}{\gamma_n + a\gamma_{n+1}} = (1-\gamma_{n+2})\left(\frac{\gamma_{n+2}}{b} - 1\right).$$

Hence we readily obtain (20), which in turn confirms (18). Therefore, the claim is valid by induction.

Claim 2. *The sequences $\{\gamma_{2n}\}_{n=0}^{+\infty}$ and $\{\gamma_{2n-1}\}_{n=0}^{+\infty}$ converge to u_2^*, respectively.*

From Claim 1, the sequence $\{\gamma_{2n}\}_{n=0}^{+\infty}$ is decreasing with u_2^* as the lower bound and the sequence $\{\gamma_{2n-1}\}_{n=0}^{+\infty}$ is increasing with u_2^* as the upper bound, then there exist two positive constants γ^* and γ_*, with $\gamma^* \ge u_2^* \ge \gamma_*$, such that $\gamma_{2n} \to \gamma^*$ and $\gamma_{2n-1} \to \gamma_*$ as $n \to +\infty$. Note that

$$|\gamma^* - \gamma_*| \le |\gamma^* - \gamma_{2n}| + |\gamma_{2n} - \gamma_{2n-1}| + |\gamma_{2n-1} - \gamma_*|,$$

it is sufficient to prove that the sequence $\gamma_{2n} - \gamma_{2n-1} \to 0$ as $n \to +\infty$. Since

$$\sup_{(1+b)/2 \le x, y \le 1} \left\{ \frac{\partial}{\partial x}\left(\frac{x}{y+ax}\right), \frac{\partial}{\partial y}\left(\frac{x}{y+ax}\right) \right\} \le \frac{4}{(1+a)^2(1+b)^2},$$

utilizing the mean value theorem allows us to conclude that for all $n \in \mathbb{N}$

$$
\begin{aligned}
&\frac{b\gamma_{2n-2}}{\gamma_{2n-3}+a\gamma_{2n-2}}-\frac{b\gamma_{2n-1}}{\gamma_{2n-2}+a\gamma_{2n-1}}\\
\le&\frac{4b}{(1+a)^2(1+b)^2}\left[(\gamma_{2n-2}-\gamma_{2n-3})+(\gamma_{2n-2}-\gamma_{2n-1})\right]\\
<&\frac{8b}{(1+a)^2(1+b)^2}(\gamma_{2n-2}-\gamma_{2n-3})\\
<&\frac{2}{(1+a)^2}(\gamma_{2n-2}-\gamma_{2n-3})
\end{aligned}
$$

owing to $\gamma_{2n-3}<\gamma_{2n-1}$. Then a direct calculation gives

$$
\begin{aligned}
\gamma_{2n}-\gamma_{2n-1}&=\frac{1}{2}\left(\sqrt{(1-b)^2-\frac{4mb\gamma_{2n-1}}{\gamma_{2n-2}+a\gamma_{2n-1}}}-\sqrt{(1-b)^2-\frac{4mb\gamma_{2n-2}}{\gamma_{2n-3}+a\gamma_{2n-2}}}\right)\\
&=\frac{2m\left(\frac{b\gamma_{2n-2}}{\gamma_{2n-3}+a\gamma_{2n-2}}-\frac{b\gamma_{2n-1}}{\gamma_{2n-2}+a\gamma_{2n-1}}\right)}{\sqrt{(1-b)^2-\frac{4mb\gamma_{2n-1}}{\gamma_{2n-2}+a\gamma_{n2-1}}}+\sqrt{(1-b)^2-\frac{4mb\gamma_{2n-2}}{\gamma_{2n-3}+a\gamma_{2n-2}}}}\\
&\le\frac{2m\left(\frac{b\gamma_{2n-2}}{\gamma_{2n-3}+a\gamma_{2n-2}}-\frac{b\gamma_{2n-1}}{\gamma_{2n-2}+a\gamma_{2n-1}}\right)}{\sqrt{(1-b)^2-\frac{4mb\gamma_{2n-1}}{\gamma_{2n-2}+a\gamma_{2n-1}}}}\\
&\le\frac{\frac{4m}{(1+a)^2}}{\sqrt{(1-b)^2-\frac{4mb}{1+a}}}(\gamma_{2n-2}-\gamma_{2n-3})\\
&:=\rho(\gamma_{2n-2}-\gamma_{2n-3}).
\end{aligned}
$$

If $\rho\in(0,1)$, that is

$$
m<\frac{(1+a)^3}{8}\left(\sqrt{b^2+4\left(\frac{1-b}{1+a}\right)^2}-b\right),
$$

then one can lightly verify $\gamma_{2n}-\gamma_{2n-1}\to 0$ as $n\to+\infty$. Hence, the claim is valid.

Now, we show that $(\phi,\psi)(+\infty)=(u_2^*,u_2^*)$. It follows from Lemma 3.4 that

$$
\gamma_{-1}<\phi_-\le\psi_-\le\psi_+\le\phi_+<\gamma_0.
$$

According to Claim 2, one can see that $(\phi,\psi)(+\infty)=(u_2^*,u_2^*)$ holds as long as

$$
\gamma_{2n-1}<\phi_-\le\psi_-\le\psi_+\le\phi_+<\gamma_{2n}\text{ for all } n\in\mathbb{N}.
$$

Hence we define

$$
n_0=\sup\left\{n\in\mathbb{N}|\gamma_{2n-1}<\phi_-\le\psi_-\le\psi_+\le\phi_+<\gamma_{2n}\right\},
$$

and we will prove $n_0=+\infty$ by contradiction, assuming initially that n_0 is finite. If that, from the definition of n_0, it holds that either

$$
\gamma_{2n_0-1}=\phi_-\le\psi_-\le\psi_+\le\phi_+<\gamma_{2n}
$$

or

$$
\gamma_{2n_0-1}<\phi_-\le\psi_-\le\psi_+\le\phi_+=\gamma_{2n}.
$$

Firstly, we assume that $\phi_-=\gamma_{2n_0-1}$. If $\phi(\xi)$ is eventually monotone, then $\phi(+\infty)=\gamma_{2n_0-1}$ and

$$
\int_0^{+\infty}\phi'(\xi)d\xi=\gamma_{2n_0-1}-\phi(0)<\infty.
$$

Note that $\limsup_{\xi\to+\infty}\phi'(\xi)=0$ if $\phi'(\xi)\le 0$ for $\xi\gg 1$ or $\liminf_{\xi\to+\infty}\phi'(\xi)=0$ if $\phi'(\xi)\ge 0$ for $\xi\gg 1$, then we can find a sequence $\{\xi_n\}_{n=0}^{+\infty}$, with $\xi_n\to+\infty$ as $n\to+\infty$, such that

$$
\lim_{n\to+\infty}\phi(\xi_n)=\gamma_{2n_0-1}\text{ and }\lim_{n\to+\infty}\phi'(\xi_n)=0.
$$

Integrating ϕ-equation of (4) from 0 to ξ_n, we obtain

$$
c[\phi(\xi_n)-\phi(0)]-d_1\int_0^{\xi_n}\mathcal{N}_1[\phi](\xi)d\xi=\int_0^{\xi_n}\phi(\xi)f(\phi,\psi)(\xi)d\xi. \tag{21}
$$

Hence, following a similar argument as in Lemma 3.3, the left-hand side of (21) is bounded, whereas the right-hand side of (21) remains unbounded since

$$
\begin{aligned}
\liminf_{n\to+\infty} f(\phi,\psi)(\xi_n) &\geq f\left(\gamma_{2n_0-1},\gamma_{2n_0-2}\right) \\
&> (1-\gamma_{2n_0-1})\left(\frac{\gamma_{2n_0-1}}{b}-1\right)-\frac{m\gamma_{2n_0-2}}{\gamma_{2n_0-3}+a\gamma_{2n_0-2}}=0,
\end{aligned} \tag{22}
$$

which leads a contradiction. While if $\phi(\xi)$ is oscillatory as $\xi\to+\infty$, we can find a sequence $\{\xi_n\}_{n=0}^{+\infty}$ of the minimal points of $\phi(\xi)$, with $\xi_n\to+\infty$ as $n\to+\infty$, such that

$$\lim_{n\to+\infty}\phi(\xi_n)=\gamma_{2n_0-1} \text{ and } \phi'(\xi_n)=0.$$

Deriving (21) once again leads to a contradiction with (22). Moreover, we can also similarly tackle another case $\phi_+=\gamma_{2n_0}$ since

$$
\begin{aligned}
\limsup_{n\to+\infty} f(\phi,\psi)(\xi_n) &\leq f\left(\gamma_{2n_0},\gamma_{2n_0-1}\right) \\
&< (1-\gamma_{2n_0})\left(\frac{\gamma_{2n_0}}{b}-1\right)-\frac{m\gamma_{2n_0-1}}{\gamma_{2n_0-2}+a\gamma_{2n_0-1}}=0.
\end{aligned}
$$

Hence we must have $n_0=+\infty$, which implies that $(\phi,\psi)(+\infty)=(u_2^*,u_2^*)$ and finishes this proof. □

4. The existence of traveling waves for $c=c^*$

This section is devoted to the existence of traveling waves of (2) for $c=c^*$, which depends on a limiting argument. Let us begin with the following result.

Lemma 4.1. *For $c>c^*$, the solution $(\phi,\psi)(\xi)$ of (4)–(5) satisfies*

$$\lim_{\xi\to-\infty}\frac{\psi'(\xi)}{\psi(\xi)}=\lambda\in\left\{\lambda_1,\ \lambda_2\right\},$$

where λ_1 and λ_2 are given in Lemma 1.1.

Proof. From Theorem 3.1, we have $\psi(\xi)>0$ over $\mathbb{R}$. Let

$$Z(\xi)=\frac{\psi'(\xi)}{\psi(\xi)} \text{ and } B(\xi)=g(\phi,\psi)(\xi)-d_2.$$

Note that

$$\frac{\psi(\xi-y)}{\psi(\xi)}=e^{\ln\psi(\xi-y)-\ln\psi(\xi)}=e^{\int_\xi^{\xi-y}\frac{\psi'(s)}{\psi(s)}ds}=e^{\int_\xi^{\xi-y}Z(s)ds},$$

then $\psi(\xi)$-equation gives

$$
\begin{aligned}
cZ(\xi)=&d_2\int_{\mathbb{R}}J_2(y)\left(\frac{\psi(\xi-y)}{\psi(\xi)}\right)dy-d_2+g(\phi,\psi)(\xi) \\
=&d_2\int_{\mathbb{R}}J_2(y)e^{\int_\xi^{\xi-y}Z(s)ds}dy+B(\xi).
\end{aligned}
$$

Since $(\phi,\psi)(-\infty)=(1,0)$, we have

$$B(-\infty)=g(-\infty)-d_2=s\left(1-\frac{\psi(-\infty)}{\phi(-\infty)}\right)-d_2=s-d_2.$$

By Lemma A.1, we know that $\lambda:=\lim_{\xi\to-\infty}Z(\xi)$ exists and satisfies

$$c\lambda=d_2\left(\int_{\mathbb{R}}J_2(y)e^{-\lambda y}dy-1\right)+s.$$

Following Lemma 1.1, we conclude this proof. □

Now, we present the existence result on traveling wave with $c=c^*$.

Theorem 4.1. *Under (7) and $b\in(0,1)$, (2) has a traveling wave connecting $(1,0)$ and (u_2^*,u_2^*) for $c=c^*$.*

Proof. We choose a strictly decreasing sequence $\{c_n\}_{n=0}^{+\infty}$ with $c_n\in(c^*,c^*+1)$ and $\lim_{n\to+\infty}c_n=c^*$. Thus, for each c_n, there exists a positive solution $(\phi_n,\psi_n)(\xi)$ of (4)–(5), and from the above lemma, we have $\psi_n'(\xi)>0$ for $\xi\ll-1$ since $\psi_n(\xi)>0$ over $\mathbb{R}$. Furthermore, for any $a\in\mathbb{R}$, because $(\phi_n,\psi_n)(\xi+a)$ is also the solution of (4)–(5), we assume that there exists a positive constant $\delta<\min\left\{(1+b)/8,u_1^*/2\right\}$ such that $\psi_n(0)=\delta$ and $\psi_n(\xi)\le\delta$ for $\xi<0$. It follows from Theorem 3.1 that $(1+b)/2<\phi_n(\xi)<1$ and $0<\psi_n(\xi)<1$ over $\mathbb{R}$, then $\phi_n'(\xi)$ and $\psi_n'(\xi)$ are uniformly bounded over $\mathbb{R}$, implying that $\phi_n(\xi)$ and $\psi_n(\xi)$ are equicontinuous. From *(J2)*, we have

$$\left|\frac{d}{d\xi}\int_{\mathbb{R}}J_1(\xi-y)\phi_n(y)\mathrm{d}y\right|=\left|\int_{\mathbb{R}}\frac{d}{d\xi}J_1(\xi-y)\phi_n(y)\mathrm{d}y\right|\leqslant\int_{\mathbb{R}}\left|J_1'(y)\right|dy,$$

$$\left|\frac{d}{d\xi}\int_{\mathbb{R}} J_2(\xi-y)\psi_n(y)\mathrm{d}y\right| = \left|\int_{\mathbb{R}} \frac{d}{d\xi} J_2(\xi-y)\psi_n(y)\mathrm{d}y\right| \leqslant \int_{\mathbb{R}} \left|J_2'(y)\right| dy.$$

By calculating derivative on ξ in (4), $\phi_n''(\xi)$ and $\psi_n''(\xi)$ are uniformly bounded over $\mathbb{R}$, then $\phi_n'(\xi)$ and $\psi_n'(\xi)$ are equicontinuous. Hence, by Arzela–Ascoli theorem, up to extracting a subsequence, there exist functions $\phi(\xi)$ and $\psi(\xi)$ such that $(\phi_n, \psi_n)(\xi)$ and $(\phi_n', \psi_n')(\xi)$ converge uniformly to $(\phi, \psi)(\xi)$ and $(\phi', \psi')(\xi)$ on every bounded interval and point-wise over $\mathbb{R}$. Additionally, the dominated convergence theorem yields that

$$\int_{\mathbb{R}} J_1(\xi-y)\phi_n(y)\mathrm{d}y \to \int_{\mathbb{R}} J_1(\xi-y)\phi(y)\mathrm{d}y \text{ as } n\to+\infty,$$
$$\int_{\mathbb{R}} J_2(\xi-y)\psi_n(y)\mathrm{d}y \to \int_{\mathbb{R}} J_2(\xi-y)\psi(y)\mathrm{d}y \text{ as } n\to+\infty.$$

Therefore, $(\phi,\psi)(\xi)$ is a solution of (4) with $c = c^*$ by letting $n \to +\infty$ in (4) with $(\phi,\psi)(\xi) = (\phi_n, \psi_n)(\xi)$. Simultaneously, we also have $(1+b)/2 \le \phi(\xi) \le 1$ and $0 \le \psi(\xi) \le 1$ over $\mathbb{R}$. Following the proof in Theorem 3.1, we still obtain that $(1+b)/2 < \phi(\xi) < 1$ and $0 < \psi(\xi) < 1$ over $\mathbb{R}$.

It is worth emphasizing that the proofs of Lemma 3.3, Lemma 3.4 and Theorem 3.2 are completely independent of the variable c, then $(\phi,\psi)(+\infty) = (u_2^*, u_2^*)$ still holds. Therefore, what we need to do is to prove $(\phi,\psi)(-\infty) = (1,0)$. Naturally, we define

$$\overline{\phi} = \limsup_{\xi\to-\infty} \phi(\xi), \quad \underline{\phi} = \liminf_{\xi\to-\infty} \phi(\xi),$$
$$\overline{\psi} = \limsup_{\xi\to-\infty} \psi(\xi), \quad \underline{\psi} = \liminf_{\xi\to-\infty} \psi(\xi).$$

Inspired by [18], we split our proof into the following two cases.

Case 1. $\overline{\phi} = \underline{\phi}$. In this case, we seek to demonstrate that $\psi(-\infty)$ indeed exists and is equal to 0. Actually, assume that $\underline{\psi} < \overline{\psi}$, there exist two sequences $\{x_n\}_{n=0}^{+\infty}$ and $\{y_n\}_{n=0}^{+\infty}$ satisfying x_n, $y_n \to -\infty$ as $n \to +\infty$ such that

$$\lim_{n\to+\infty} \psi(x_n) = \underline{\psi} \text{ and } \lim_{n\to+\infty} \psi(y_n) = \overline{\psi}.$$

From Lemma A.2, we have $\phi'(-\infty) = 0$ since $\phi(-\infty)$ exists. Using Lemma A.3, for any sequence $\{\tau_n\}_{n=0}^{+\infty}$ with $\tau_n \to -\infty$ as $n \to +\infty$, we get

$$\lim_{n\to+\infty} \mathcal{N}_1[\phi](\tau_n) = 0.$$

Selecting $\{\tau_n\}_{n=0}^{+\infty}$ as $\{x_n\}_{n=0}^{+\infty}$ and $\{y_n\}_{n=0}^{+\infty}$ respectively, and letting $n \to +\infty$, we arrive at

$$\phi(-\infty) f\left(\phi(-\infty), \underline{\psi}\right) = 0 \text{ and } \phi(-\infty) f\left(\phi(-\infty), \overline{\psi}\right) = 0.$$

Since $\phi(\xi) \ge (1+b)/2$ over $\mathbb{R}$, we must have

$$f\left(\phi(-\infty), \underline{\psi}\right) = f\left(\phi(-\infty), \overline{\psi}\right) = 0.$$

From the expression of $f(\phi,\psi)$, automatically, we get

$$\frac{m\underline{\psi}}{\phi(-\infty) + a\underline{\psi}} = \frac{m\overline{\psi}}{\phi(-\infty) + a\overline{\psi}},$$

which yields that $\underline{\psi} = \overline{\psi}$ and $\psi(-\infty)$ exists. From Lemma A.2, we have $\psi'(-\infty) = 0$. Similarly, for any sequence $\{\tau_n\}_{n=0}^{+\infty}$ with $\tau_n \to -\infty$ as $n \to +\infty$,

$$\lim_{n\to+\infty} \mathcal{N}_2[\psi](\tau_n) = 0.$$

Hence, we have

$$\phi(-\infty) f\left(\phi(-\infty), \psi(-\infty)\right) = 0 \text{ and } \psi(-\infty) g\left(\phi(-\infty), \psi(-\infty)\right) = 0.$$

Thus, $(\phi(-\infty), \psi(-\infty))$ might converge to

$$(b,0),\ (1,0),\ (u_1^*, u_1^*) \text{ or } (u_2^*, u_2^*).$$

Since $\delta < u_1^*/2$ and $\phi(\xi) \ge (1+b)/2$, we have $(\phi(-\infty), \psi(-\infty)) = (1,0)$.

Case 2. $\overline{\phi} \ne \underline{\phi}$. We claim that $\psi(-\infty) = 0$. Otherwise, there exist $\zeta \in (0,\delta)$ such that

$$\limsup_{\xi\to-\infty} \psi(\xi) = \zeta. \tag{23}$$

Up to extracting a subsequence, there exists a sequence $\{\ell_n\}_{n=0}^{+\infty}$, with $\ell_n \to -\infty$ as $n \to +\infty$, such that

$$\psi(\ell_n) > \frac{\zeta}{2} \text{ for all } n.$$

From the uniform continuity of $\psi(\xi)$, we have for an appropriate $\epsilon_1 > 0$

$$\psi(\ell_n+\xi) > \frac{\zeta}{4} \text{ for } \xi \in (-\epsilon_1, \epsilon_1).$$

Now, we consider initial value problem

$$\begin{cases} \varphi_t = d_2\mathcal{N}_2[\varphi](x,t) + s\varphi\left(1 - \dfrac{2\varphi}{1+b}\right), \\ \varphi(x,0) = \varphi(x), \end{cases}$$

where $\varphi(x)$ satisfies the following conditions:

(1) $\varphi(x)$ is uniformly continuous on x,

(2) $\varphi(x) = \zeta/4$ for $x \in [-\epsilon_1/2, \epsilon_1/2]$,

(3) $\varphi(x)$ is decreasing for $x \in [\epsilon_1/2, \epsilon_1]$ and increasing for $x \in [-\epsilon_1/2, -\epsilon_1]$,

(4) $\varphi(x) = 0$ for $|x| > \epsilon_1$.

Spreading speed theory [35] gives for $c \in (0, c^*)$

$$\liminf_{t\to+\infty} \inf_{|x|<ct} \varphi(x,t) = \frac{1+b}{2}.$$

Thanks to $\delta < (1+b)/8$, there exists a constant $T > 0$ such that for $c \in (0, c^*)$

$$\inf_{|x|<ct} \varphi(x,t) > 2\delta.$$

For the above T, we further choose two subsequences $\{\ell_{1n}\}_{n=0}^{+\infty}$ and $\{\ell_{2n}\}_{n=0}^{+\infty}$ satisfying for all n

$$\ell_{1n} - \ell_{2n} > c^*T, \ \psi(\ell_{1n}) > \frac{\zeta}{2} \text{ and } \psi(\ell_{2n}) > \frac{\zeta}{2}.$$

From ψ-equation, the function $w(x,t) := \psi(x + c^*t + \ell_{2n})$ satisfies

$$\begin{cases} w_t \ge d_2\mathcal{N}_2[w](x,t) + sw\left(1 - \dfrac{2w}{1+b}\right), \\ w(x,0) = \psi(x+\ell_{2n}). \end{cases}$$

By the comparison principle [35], we have for $c \in (0, c^*)$

$$\liminf_{t\to+\infty} \inf_{|x|<ct} w(x,t) \ge \liminf_{t\to+\infty} \inf_{|x|<ct} \varphi(x,t) = (1+b)/2.$$

Now, we fix $x = 0$ and $t = (\ell_{1n} - \ell_{2n})/c^*$, obviously, $|x| < ct$ for $c \in (0, c^*)$. Then for $t > T$

$$w(0,t) = w(0, (\ell_{1n} - \ell_{2n})/c^*) = \psi(\ell_{1n}) > \varphi(0,t) > 2\delta > \zeta$$

by the choice of ζ. Hence we obtain

$$\limsup_{\xi\to-\infty} \psi(\xi) > \zeta,$$

which contradicts to (23). Therefore, we have $\psi(-\infty) = 0$. Under the condition $\overline{\phi} \neq \underline{\phi}$, there exist sequences $\{x_n\}_{n=0}^{+\infty}$ and $\{y_n\}_{n=0}^{+\infty}$, with x_n and $y_n \to -\infty$ as $n \to +\infty$, such that

$$\lim_{n\to+\infty} \phi(x_n) = \underline{\phi} \text{ and } \phi'(x_n) = 0,$$
$$\lim_{n\to+\infty} \phi(y_n) = \overline{\phi} \text{ and } \phi'(y_n) = 0.$$

From Lemma A.3, taking $\xi = x_n$ or $\xi = y_n$ in ϕ-equation, and letting $n \to +\infty$, we have

$$\underline{\phi}\left(1 - \underline{\phi}\right)\left(\frac{\underline{\phi}}{b} - 1\right) \le 0,$$
$$\overline{\phi}\left(1 - \overline{\phi}\right)\left(\frac{\overline{\phi}}{b} - 1\right) \ge 0.$$

Since $\overline{\phi} > \underline{\phi} \ge (1+b)/2$, we have $\underline{\phi} < \overline{\phi} \le 1 \le \underline{\phi}$, which suggests that $\overline{\phi} \neq \underline{\phi}$ cannot occur.

Therefore, this proof is finalized by combining **Case 1** with **Case 2.** □

Finally, let us finish the proof of Theorem 1.1 by demonstrating the non-existence of traveling waves with $0 < c < c^*$.

Theorem 4.2. *For $0 < c < c^*$, (2) has no traveling waves connecting $(1,0)$ and (u_2^*, u_2^*).*

Proof. By contradiction, we assume that there exists a positive solution $(\phi,\psi)(\xi)$ of (4)–(5) for given $0 < c_0 < c^*$. Through the boundary conditions (5), we have $\phi(\xi) \ge 1/K$ over $\mathbb{R}$ for some $K \gg 1$. Let $w(x,t) := \psi(x + c_0 t)$, from ψ-equation, we get

$$\begin{cases} w_t \geq d_2\mathcal{N}_2[w](x,t) + sw(x,t)\left(1 - Kw(x,t)\right), \\ w(0,x) = \psi(x). \end{cases}$$

Spreading speed theory and the comparison principle [35] give

$$\lim_{t\to+\infty} \inf_{2|x|=(c_0+c^*)t} w(x,t) \geq \frac{1}{K}.$$

As $2x = -(c_0 + c^*)t$, we have

$$w(x,t) = \psi(x + c_0 t) = \psi\left(-\frac{(c_0+c^*)t}{2} + c_0 t\right) = \psi\left(\frac{(c_0-c^*)t}{2}\right) \geq \frac{1}{K},$$

which contradicts to $\psi(-\infty) = 0$. Therefore, we complete the proof. □

Acknowledgments

We are grateful to the anonymous reviewers for their constructive feedback, which has significantly improved the quality of this manuscript.

Appendix

The appendix contains crucial lemmas used throughout this paper.

Lemma A.1 (*[36]*). *Assume $c > 0$ and $B(\xi)$ is a continuous function with $B(\pm\infty) := \lim_{\xi\to\pm\infty} B(\xi)$. Let $Z(\xi)$ be a measurable function satisfying*

$$cZ(\xi) = \int_{\mathbb{R}} J_i(y) e^{\int_\xi^{\xi-y} Z(s)ds} dy + B(\xi) \text{ in } \mathbb{R}, \quad i = 1,2.$$

Then, $Z(\xi)$ is uniformly continuous and bounded. Moreover, $\mu_\pm := \lim_{\xi\to\pm\infty} Z(\xi)$ exist and are real roots of the characteristic equation

$$c\mu_\pm = \int_{\mathbb{R}} J_i(y) e^{-(\mu_\pm)y} dy + B(\pm\infty), \quad i = 1,2.$$

Lemma A.2 (*[37]*). *Assume that $w(\xi) \in C^1(b,+\infty)$ and $\lim_{\xi\to+\infty} w(\xi)$ exists. If $w'(\xi)$ is uniformly continuous, then $\lim_{\xi\to+\infty} w'(\xi) = 0$.*

Lemma A.3 (*[38]*). *Assume that $J(\xi) \geq 0$ and $\int_{\mathbb{R}} J(\xi)d\xi = 1$, and $\omega(\xi)$ is a nonnegative bounded continuous function on $\mathbb{R}$. Then we have*

$$\liminf_{\xi\to\infty} \int_{\mathbb{R}} J(y)\omega(\xi - y)dy \geq \liminf_{\xi\to\infty} \omega(\xi) := \omega^-,$$
$$\limsup_{\xi\to\infty} \int_{\mathbb{R}} J(y)\omega(\xi - y)dy \leq \limsup_{\xi\to\infty} \omega(\xi) := \omega^+.$$

In particular, if $\omega(\infty)$ exists, that is, $\omega^- = \omega^+ = \omega(\infty)$, then

$$\lim_{\xi\to\infty} \int_{\mathbb{R}} J(y)\omega(\xi - y)dy = \omega(\infty).$$

Data availability

The data that supports the findings of this study are available within the article.

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