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Traveling waves of modified Leslie–Gower predator–prey systems

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This paper focuses on the spreading phenomena within modified Leslie–Gower reaction–diffusion predator–prey systems. Our main objective is to investigate the existence of two distinct types of traveling waves. Specifically, with the aid of the upper and lower solution methods, we establish the existence of traveling waves connecting the prey-present state and the coexistence state or the prey-present state and the prey-free state by constructing different and appropriate Lyapunov functions. Moreover, for traveling wave connecting the prey-present state and the prey-free state, more information about the monotonicity of the wave profile can be obtained by analyzing its asymptotic behavior at negative infinity. Finally, our results are applied to modified Leslie–Gower system with Holling-II type functional response or Lotka–Volterra type functional response, and a novel Lyapunov function is constructed for the latter, which further enhances our results. Meanwhile, some numerical simulations are carried out to support our results.

Keywords: Modified Leslie–Gower system; traveling waves; LaSalle's invariance principle.

1. Introduction

Due to the complexity of ecosystems, long-term coexisting species frequently exhibit diversified interspecific interactions, such as cooperation, competition, and predation. One of the most well-known systems is the Lotka–Volterra predator–prey system, developed by Lotka [29] and Volterra [37] in the 1920s, and there has been a substantial amount of research dedicated to studying predator–prey systems, seeing [1, 3, 9, 22, 31, 32]. However, the majority of biological species live in spatially

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heterogeneous natural habitats, and it is fair to anticipate that habitat variability affects the population density. Consequently, the response terms should be expanded to include diffusion.

Our interest lies in the alteration of ecological processes following the introduction of alien species into new habitats. One way to study this subject is to look at the so-called traveling waves, seeing [13–15, 24–26], another way is to characterize the spreading speed of the predator by solving Cauchy problem, seeing [4, 5, 11]. It should be emphasized that the functional response, the rate of prey consumption by an average predator, and the predator growth are identical in most two species predator–prey systems. For example,

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \Delta u(x,t) + g(u(x,t)) - f(u(x,t))v(x,t), \\ \dfrac{\partial v(x,t)}{\partial t} = d\Delta v(x,t) + (\beta f(u(x,t)) - d)v(x,t). \end{cases}$$

Some scholars have also studied predator–prey systems whose predator growth function differs from the functional response. For example, Leslie in [27, 28] proposed a predator–prey system in which the predator's environmental carrying capacity is proportional to the abundance of the prey

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \Delta u(x,t) + u(x,t)\left(1 - u(x,t) - \dfrac{av(x,t)}{u(x,t)+e_1}\right), \\ \dfrac{\partial v(x,t)}{\partial t} = d\Delta v(x,t) + sv(x,t)\left(1 - \dfrac{v(x,t)}{u(x,t)}\right), \end{cases}$$

where the term $v(x,t)/u(x,t)$ is called the Leslie–Gower term. Recently, the existence of traveling waves of the above system has attracted increasing interest, including Lotka–Volterra type, Holling-III type and Beddington–DeAngelis functional responses, and so on, seeing [2, 10, 12, 38, 41].

Based on the above system, the authors [6, 7] proposed a modified Leslie–Gower predator–prey system, incorporating the positive constant μ into the Leslie–Gower term. Such predators are called *generalist predators*[20] since they explore other alternative food sources in the absence of their favorite prey. More precisely,

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \Delta u(x,t) + u(x,t)\left(1 - u(x,t) - \dfrac{av(x,t)}{u(x,t)+e_1}\right), \\ \dfrac{\partial v(x,t)}{\partial t} = d\Delta v(x,t) + sv(x,t)\left(1 - \dfrac{v(x,t)}{u(x,t)+\mu}\right), \end{cases} \tag{1.1}$$

where parameters a, s, e_1 and μ are positive, $u(x,t)$ and $v(x,t)$ represent the population densities of two species, respectively, and d denotes the ratio of the diffusion of the predator to that of the prey. For more detailed descriptions, one can refer to [6, 7].

When the prey is stationary, Tian *et al.* in [34, 35] studied traveling waves connecting two appropriate equilibria by the shooting argument and the upper and lower solution method, respectively. However, when the prey diffuses more slowly than the predator, Hsu and Lin [23] obtained the existence of traveling waves connecting $(0, 0)$ and the coexistence state using the upper and lower solution method with the aid of contracting rectangles. For system (1.1), to the best of the author's knowledge, there are no results on traveling waves connecting the prey-present state and the coexistence state or the prey-present state and the prey-free state, even for a modified Leslie–Gower predator–prey system with Lotka–Volterra type [17, 19]. Based on these, we consider a generalized Leslie–Gower system

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \Delta u(x,t) + f(u(x,t))(p(u(x,t)) - v(x,t)), \\ \dfrac{\partial v(x,t)}{\partial t} = d\Delta v(x,t) + sv(x,t)\left(1 - \dfrac{v(x,t)}{u(x,t)h(u(x,t)) + \mu}\right), \end{cases} \tag{1.2}$$

where parameters d, s, μ are positive constants. In this paper, we assume that $f(u)$, $p(u)$ and $h(u)$ are C^1 functions satisfying the following assumption:

Assumption 1.1.

(H1) $f(0) = 0$ and $f'(u) > 0$ for $u \in [0, +\infty)$.
(H2) $p(1) = 0$ and $p'(u) < 0$ for $u \in (0, +\infty)$.
(H3) $h(0) > 0$ and $h'(u) \geq 0$ for $u \in [0, +\infty)$.

If the prey follows the logistic growth pattern, then some classical examples fulfilling Assumption 1.1 are as follows:

(1) (1.2) with Holling-I type functional response $f(u)$: for $a > 0$,

$$f(u) = au, \quad p(u) = \frac{(1-u)}{a} \quad \text{and} \quad h(u) = 1.$$

(2) (1.2) with Holling-II type functional response $f(u)$: for $a > 0$ and $e_1 \geq 1$,

$$f(u) = \frac{au}{u + e_1}, \quad p(u) = \frac{(1-u)(u+e_1)}{a} \quad \text{and} \quad h(u) = 1.$$

(3) (1.2) with Ivlev type functional response $f(u)$: for $a > 0$ and $2 > m > 0$,

$$f(u) = a(1 - e^{-mu}), \quad p(u) = \frac{u(1-u)}{a(1-e^{-mu})} \quad \text{and} \quad h(u) = 1.$$

In view of Assumption 1.1, we consider the corresponding kinetic system, there are always three boundary equilibria $(0, 0)$, $(1, 0)$ and $(0, \mu)$. Clearly, $(1, 0)$ is unstable. Moreover, one can readily verify that $(0, \mu)$ is stable if $\mu > p(0)$, whereas $(0, \mu)$ is unstable if $\mu < p(0)$. On the other hand, we also conclude that the function $Q(u) = p(u) - uh(u) - \mu$ satisfies $Q(1) < 0$, $Q(0) > 0$ and $Q'(u) < 0$ for $u \in (0, 1)$ if $p(0) > \mu$, implying that system (1.2) has a unique positive equilibrium $(u^*, v^*) := (u^*, u^*h(u^*) + \mu)$ with $u^* \in (0, 1)$. In order to have traveling waves connecting $(1,0)$ and (u^*,v^*), we also require the stability property on (u^*,v^*). Let J_* be the linearized matrix at (u^*,v^*) of the corresponding kinetic system, that is

$$J_* = \begin{pmatrix} f(u^*)p'(u^*) & -f(u^*) \\ s(u^*h'(u^*)+h(u^*)) & -s \end{pmatrix}.$$

A straightforward calculation yields that $trace(J_*) = f(u^*)p'(u^*) - s < 0$ and $det(J_*) = -sf(u^*)Q'(u^*) > 0$, then (u^*,v^*) is a stable equilibrium. Consequently, we may expect traveling waves connecting $(1,0)$ and E of system (1.2), where

$$E = \begin{cases} (0,\mu) & \text{if } \mu > p(0), \\ (u^*,v^*) & \text{if } \mu < p(0). \end{cases}$$

We are interested in the spreading phenomena of system (1.2). First of all, let us focus on the spreading speed of $v(x,t)$. Constant s^* is called *the spreading speed* of species $w(x,t)$ if

$$\lim_{t\to\infty}\left\{\sup_{|x|>(s^*+\epsilon)t} w(x,t)\right\} = 0 \quad \text{for } \epsilon > 0,$$

$$\liminf_{t\to\infty}\left\{\inf_{|x|<(s^*-\epsilon)t} w(x,t)\right\} > 0 \quad \text{for } \epsilon \in (0,s^*).$$

Assuming that $0 \le u(x,0) \le 1$ and $0 \le v(x,0) \le h(1)+\mu$ are continuous functions with nonempty compact support, we can claim, by the theory of reaction–diffusion system, that $[0,1]\times[0,h(1)+\mu]$ is an invariant region of system (1.2). Hence, we get that $(u(x,t),v(x,t)) \in [0,1]\times[0,h(1)+\mu]$ for $x\in\mathbb{R}$ and $t>0$. Meanwhile, one can directly observe that

$$\frac{\partial v(x,t)}{\partial t} \le d\Delta v(x,t) + sv(x,t)\left(1 - \frac{v(x,t)}{h(1)+\mu}\right)$$

and

$$\frac{\partial v(x,t)}{\partial t} \ge d\Delta v(x,t) + sv(x,t)\left(1 - \frac{v(x,t)}{\mu}\right),$$

thus, the spreading theory and the comparison principle of [4, 5] give

$$\lim_{t\to\infty}\left\{\sup_{|x|>(2\sqrt{ds}+\epsilon)t} v(x,t)\right\} = 0 \quad \text{for } \epsilon > 0,$$

$$\liminf_{t\to\infty}\left\{\inf_{|x|<(2\sqrt{ds}-\epsilon)t} v(x,t)\right\} > \mu \quad \text{for } \epsilon \in (0,2\sqrt{ds}),$$

implying that the spreading speed of $v(x,t)$ is $c^* = 2\sqrt{ds}$. Therefore, this paper focuses on establishing the existence of traveling waves that connect $(1,0)$ and E of system (1.2). A positive solution is called a traveling wave of system (1.2) if it has the form $(u(x,t), v(x,t)) = (\tilde{u}(z), \tilde{v}(z))$, $z = x + ct$, where c is wave speed. With the tilde removed, $(u(z), v(z))$ should satisfy

$$\begin{cases} u''(z) - cu'(z) + f(u(z))(p(u(z)) - v(z)) = 0, \\ dv''(z) - cv'(z) + sv(z)\left(1 - \dfrac{v(z)}{u(z)h(u(z)) + \mu}\right) = 0, \end{cases} \tag{1.3}$$

with the boundary conditions

$$(u(-\infty), v(-\infty)) = (1, 0), \quad (u(+\infty), v(+\infty)) = E. \tag{1.4}$$

Such a solution is referred to as a *traveling wave* connecting $(1,0)$ and E of system (1.2). In addition, one that just meets the left-hand tail limit is referred to as a *semi-traveling wave* of system (1.2). In the sequence, (semi-)traveling waves as mentioned are the above waves. Moreover, we assume that $q(u) = uh(u) + \mu$ in this paper for convenience. Clearly, $q(0) = \mu$, $q(1) = h(1) + \mu$ and $q(u)$ is a monotone increasing function in $[0, +\infty)$.

We are now in this position to state the existence results on traveling waves. The first theorem sets out a necessary and sufficient condition for the existence of semi-traveling waves.

Theorem 1.1. *Assume that* (H1)–(H3) *hold. System* (1.2) *has a semi-traveling wave with wave speed* c *if and only if* $c \geq c^* := 2\sqrt{ds}$.

The subsequent theorem establishes the existence of traveling waves connecting $(1,0)$ and (u^*, v^*) of a class of modified Leslie–Gower predator–prey systems, that is $h(u) = 1$.

Theorem 1.2. *Assume that* (H1) *and* (H2) *hold and* $h(u) = 1$. *For* $c \geq c^*$, *if* $p(0) > \mu$ *and* $(u^* + \mu)f(u) - (u - u^*)(u + \mu)f'(u) > 0$ *for* $u \in (0,1)$, *then system* (1.2) *has a traveling wave connecting* $(1,0)$ *and* (u^*, v^*).

Finally, we present the existence of traveling waves connecting $(1,0)$ and $(0,\mu)$. Prior to this, we define sets

$$\mathcal{A} = \{v \in C^2(\mathbb{R}) \,|\, 0 < v(z) < \mu \text{ and } v'(z) > 0 \text{ over } \mathbb{R}\},$$

$$\mathcal{B} = \left\{ v \in C^2(\mathbb{R}) \,\middle|\, \begin{array}{l} \exists\, z_v \text{ such that } v(z_v) = \mu, v'(z) > 0 \text{ in } (-\infty, z_v) \\ \text{and } v(z) > \mu \text{ in } (z_v, +\infty) \end{array} \right\}.$$

Theorem 1.3. *Assume that* (H1)–(H3) *hold. For* $c \geq c^*$, *if*

$$f(u) = ug(u) \quad \textit{and} \quad p(u) = (1 - u)/g(u),$$

where $g(u)$ *satisfies*

$$\min_{u \in [0,1]} g(u) > \frac{h(1)(h(1) + \mu)}{\mu^2 h(0)},$$

then system (1.2) *has a traveling wave* $(u(z), v(z))$ *connecting* $(1,0)$ *and* $(0,\mu)$, *where* $v(z) \in \mathcal{A} \cup \mathcal{B}$ *and* $u'(z) < 0$ *over* $\mathbb{R}$.

This paper is organized as follows. In Sec. 2, we introduce some preliminary knowledge. In Sec. 3, using the upper and lower solution method and the unstable manifold theorem, we derive a necessary and sufficient condition for the existence of semi-traveling waves. In Sec. 4, we establish the existence of traveling waves, and with regard to traveling waves connecting $(1,0)$ and $(0,\mu)$, we further gain additional monotonicity information on the wave profile by analyzing its asymptotic behavior at $z=-\infty$. In Sec. 5, we apply Theorems 1.2 and 1.3 to modified Leslie–Gower system with Holling-II type or Lotka–Volterra type, and a novel Lyapunov function is constructed for the latter, which further enhances our results, meanwhile, the numerical simulations of traveling waves are carried out to support our results. Discussions are presented in Sec. 6. In the appendix, we explain how to calculate the upper solution and the lower solution.

2. Preliminary

In this section, we will introduce the upper and lower solution method, which actually provides a constructive approach for producing traveling waves.

Let us begin with the definitions of the upper solution and the lower solution of system (1.3). Define sets X and X_0 by

$$X = \{\Phi(z) = (u(z), v(z)) \,|\, \Phi(z) \in C(\mathbb{R}, \mathbb{R}^2)\},$$

$$X_0 = \{\Phi(z) \in X \,|\, 0 \le u(z) \le 1, 0 \le v(z) \le q(1), z \in \mathbb{R}\}.$$

Definition 2.1. The function pairs $(\overline{u}(z), \overline{v}(z))$ and $(\underline{u}(z), \underline{v}(z))$ in X_0 are called the upper solution and the lower solution of system (1.3), respectively, if they satisfy

(a) $\underline{u}(z) \le \overline{u}(z)$, $\underline{v}(z) \le \overline{v}(z)$ for $z \in \mathbb{R}$.
(b) There is a finite set Ω such that for $z \in \mathbb{R}\backslash\Omega$

$$\mathcal{U}(\overline{u}, \underline{v}) = \overline{u}''(z) - c\overline{u}'(z) + f(\overline{u}(z))(p(\overline{u}(z)) - \underline{v}(z)) \le 0,$$

$$\mathcal{U}(\underline{u}, \overline{v}) = \underline{u}''(z) - c\underline{u}'(z) + f(\underline{u}(z))(p(\underline{u}(z)) - \overline{v}(z)) \ge 0,$$

$$\mathcal{V}(\overline{u}, \overline{v}) = d\overline{v}''(z) - c\overline{v}'(z) + s\overline{v}(z)\left(1 - \frac{\overline{v}(z)}{q(\overline{u}(z))}\right) \le 0,$$

$$\mathcal{V}(\underline{u}, \underline{v}) = d\underline{v}''(z) - c\underline{v}'(z) + s\underline{v}(z)\left(1 - \frac{\underline{v}(z)}{q(\underline{u}(z))}\right) \ge 0.$$

To apply Schauder's fixed point theorem, we define the Banach space

$$B_\rho(\mathbb{R}, \mathbb{R}^2) = \{\phi(z) = (u(z), v(z)) \in C(\mathbb{R}, \mathbb{R}^2) : |\phi(z)|_\rho < +\infty\}$$

with the norm

$$|\phi(z)|_\rho = \max\left\{\sup_{z\in\mathbb{R}} |u(z)|e^{-\rho|z|},\ \sup_{z\in\mathbb{R}} |v(z)|e^{-\rho|z|}\right\}$$

for small positive constant ρ. Then we will seek traveling waves in wave profile set

$$\Sigma = \{(u,v) \in C(\mathbb{R},\mathbb{R}^2) : \underline{u}(z) \le u(z) \le \overline{u}(z), \underline{v}(z) \le v(z) \le \overline{v}(z)\}.$$

Obviously, Σ is bounded, closed and convex in $C(\mathbb{R},\mathbb{R}^2)$. Furthermore, we define

$$\begin{aligned} F(u,v) &= \tau u + f(u)(p(u) - v), \\ G(u,v) &= \tau v + sv\left(1 - \frac{v}{q(u)}\right), \end{aligned}$$

where τ satisfies

$$\tau > \max\left\{\max_{u\in[0,1]}\{f(1)|p'(u)| + q(1)|f'(u)|\}, s\left(2\frac{q(1)}{q(0)} - 1\right)\right\}.$$

At the moment, one can verify that for $(u,v) \in [0,1] \times [0,q(1)]$

$$\frac{\partial F(u,v)}{\partial u} \ge 0, \quad \frac{\partial F(u,v)}{\partial v} \le 0, \quad \frac{\partial G(u,v)}{\partial u} \ge 0, \quad \frac{\partial G(u,v)}{\partial v} \ge 0.$$

Thus, system (1.3) can be rewritten as

$$\begin{cases} u'' - cu' - \tau u + F(u,v) = 0, \\ dv'' - cv' - \tau v + G(u,v) = 0. \end{cases}$$

Next, we define the operator $P = (P_1, P_2) : \Sigma \to C(\mathbb{R},\mathbb{R}^2)$ by

$$\begin{aligned} P_1(u,v)(z) &= \frac{1}{\nu_1^+ - \nu_1^-}\left\{\int_{-\infty}^{z} e^{\nu_1^-(z-y)} + \int_{z}^{\infty} e^{\nu_1^+(z-y)}\right\} F(u,v)(y)dy, \\ P_2(u,v)(z) &= \frac{1}{d(\nu_2^+ - \nu_2^-)}\left\{\int_{-\infty}^{z} e^{\nu_2^-(z-y)} + \int_{z}^{\infty} e^{\nu_2^+(z-y)}\right\} G(u,v)(y)dy, \end{aligned}$$

where

$$\nu_1^{\pm} = \frac{1}{2}(c \pm \sqrt{c^2 + 4\tau}), \quad \nu_2^{\pm} = \frac{1}{2d}(c \pm \sqrt{c^2 + 4d\tau}).$$

It is straightforward to verify that a fixed point of operator P is identical to the solution of system (1.3). Then a standard argument similar to [30, 33, 39, 40] gives that $P(\Sigma) \subseteq \Sigma$, P is continuous and compact with respect to the norm in $B_\rho(\mathbb{R},\mathbb{R}^2)$, thus, solution of system (1.3) can be obtained using Schauder's fixed point theorem. As a consequence, we conclude the following lemma.

Lemma 2.2. *Assume that system* (1.3) *has the upper solution* $(\overline{u}(z), \overline{v}(z))$ *and the lower solution* $(\underline{u}(z), \underline{v}(z))$ *satisfying for* $z_i \in \Omega$,

$$\begin{aligned} \underline{u}'(z_i^-) \le \underline{u}'(z_i^+), \quad \overline{u}'(z_i^+) \le \overline{u}'(z_i^-), \\ \underline{v}'(z_i^-) \le \underline{v}'(z_i^+), \quad \overline{v}'(z_i^+) \le \overline{v}'(z_i^-). \end{aligned}$$

Then it has a solution $(u(z), v(z))$ *satisfying for* $z \in \mathbb{R}$

$$\underline{u}(z) \le u(z) \le \overline{u}(z), \quad \underline{v}(z) \le v(z) \le \overline{v}(z).$$

3. The Existence of Semi-traveling Waves

In this section, we employ the upper and lower solution method to establish a positive solution of system (1.3) that meets the left-hand tail limit in (1.4), that is, semi-traveling wave. We further investigate the asymptotic behavior of such wave at $z = -\infty$ by using the unstable manifold theorem, and then provide the sufficient and necessary condition for the existence of semi-traveling waves.

To construct appropriate the upper solution and lower solution, let

$$P(c, \lambda) = d\lambda^2 - c\lambda + s,$$

which yields two roots λ_1 and λ_2 as follows:

$$\lambda_1 = \frac{1}{2d}(c - \sqrt{c^2 - 4ds}), \quad \lambda_2 = \frac{1}{2d}(c + \sqrt{c^2 - 4ds}). \tag{3.1}$$

Case 1. Assume that $c > c^* = 2\sqrt{ds}$, then $0 < \lambda_1 < \lambda_2$. For given constants

$$\begin{aligned}
&0 < \beta < \min\{c, \lambda_1\},\\
&0 < \varepsilon < \min\{\lambda_1,\ \lambda_2 - \lambda_1\},\\
&\sigma > \max\left\{1, \frac{f(1)q(1)}{\beta(c-\beta)}\right\},\\
&r > \max\left\{1, \frac{-sq(1)}{P(c, \lambda_1 + \varepsilon)q(0)}\right\},
\end{aligned}$$

we define functions

$$\begin{aligned}
&\overline{u}(z) = 1, && \underline{u}(z) = \begin{cases} 1 - \sigma e^{\beta z}, & z \le z_1,\\ 0, & z > z_1, \end{cases}\\
&\overline{v}(z) = \begin{cases} q(1)e^{\lambda_1 z}, & z \le 0,\\ q(1), & z > 0, \end{cases} && \underline{v}(z) = \begin{cases} q(1)e^{\lambda_1 z}(1 - re^{\varepsilon z}), & z \le z_2,\\ 0, & z > z_2, \end{cases}
\end{aligned} \tag{3.2}$$

where $z_1 = -(1/\beta)\ln\sigma < 0$ and $z_2 = -(1/\varepsilon)\ln r < 0$ by the choice of σ and r.

Case 2. Assume that $c = c^*$, then $\lambda_1 = \lambda_2 = \lambda := c/(2d)$. For given constants

$$\begin{aligned}
&h = \lambda e^2/2,\\
&0 < \beta < \min\{c, \lambda\},\\
&\sigma > \max\left\{e^{2\beta/\lambda}, \frac{f(1)hq(1)}{(c-\beta)(\lambda-\beta)\beta e}\right\},\\
&r > \max\left\{h\sqrt{2/\lambda}, \frac{4sh^2q(1)}{dq(0)}\left(\frac{7}{2e\lambda}\right)^{7/2}\right\},
\end{aligned}$$

we define functions

$$\overline{u}(z) = 1, \qquad \underline{u}(z) = \begin{cases} 1 - \sigma e^{\beta z}, & z \le z_1, \\ 0, & z > z_1, \end{cases}$$

$$\overline{v}(z) = \begin{cases} -hq(1) z e^{\lambda z}, & z \le -2/\lambda, \\ q(1), & z > -2/\lambda, \end{cases} \quad \underline{v}(z) = \begin{cases} q(1) e^{\lambda z}(-hz - r(-z)^{1/2}), & z \le z_2, \\ 0, & z > z_2, \end{cases} \tag{3.3}$$

where $z_1 = -(1/\beta)\ln\sigma$ and $z_2 = -(r/h)^2$ with $z_1, z_2 < -2/\lambda$ by the choice of σ and r.

The next step is to verify that $(\overline{u}(z), \overline{v}(z))$ and $(\underline{u}(z), \underline{v}(z))$ described in (3.2)–(3.3) are the upper solution and the lower solution of system (1.3) for $c \ge c^*$, respectively. Note that $\underline{u}(z) \le \overline{u}(z)$, $\underline{v}(z) \le \overline{v}(z)$ over $z \in \mathbb{R}$, then it is sufficient to verify requirement (b) in Definition 2.1. Since the part of the calculation is complex and lengthy, we place it in the appendix. Combining Case 1 and Case 2 in the appendix, we immediately conclude the following lemma.

Lemma 3.1. *For $c \ge c^*$, system (1.3) has a non-negative solution $(u(z), v(z))$.*

Proof. From the appendix, $(\overline{u}(z), \overline{v}(z))$ and $(\underline{u}(z), \underline{v}(z))$ described in (3.2)–(3.3) are the upper solution and the lower solution of system (1.3) with $c \ge c^*$, respectively. Moreover, for $c > c^*$, we have

$$\underline{u}'(z_1^-) = -\beta < 0 = \underline{u}'(z_1^+), \quad \overline{v}'(0^+) = 0 < q(1)\lambda_1 = \overline{v}'(0^-),$$
$$\underline{v}'(z_2^-) = -q(1)\varepsilon e^{\lambda_1 z_2} < 0 = \underline{v}'(z_2^+).$$

While, for $c = c^*$, we have

$$\underline{u}'(z_1^-) = -\beta < 0 = \underline{u}'(z_1^+), \quad \underline{v}'(z_2^-) = -q(1)he^{\lambda z_2}/2 < 0 = \underline{v}'(z_2^+),$$
$$\overline{v}'((-2/\lambda)^+) = 0 < q(1)he^{-2} = \overline{v}'((-2/\lambda)^-).$$

Hence, we complete the proof with the aid of Lemma 2.2. □

Actually, such non-negative solution is a semi-traveling wave of system (1.2).

Theorem 3.2. *For $c \ge c^* := 2\sqrt{ds}$, system (1.3) has a solution $(u(z), v(z))$ satisfying $(u, u', v, v')(-\infty) = (1, 0, 0, 0)$ and $0 < u(z) < 1$, $0 < v(z) < q(1)$ for $z \in \mathbb{R}$.*

Proof. From Lemma 2.2, we can easily get that for $c \ge c^*$

$$0 \le \underline{u}(z) \le u(z) \le \overline{u}(z) = 1 \quad \text{and} \quad 0 \le \underline{v}(z) \le v(z) \le \overline{v}(z) \le q(1) \quad \text{for } z \in \mathbb{R}.$$

It is clear that

$$1 = \lim_{z\to-\infty} \underline{u}(z) \le \liminf_{z\to-\infty} u(z) \le \limsup_{z\to-\infty} u(z) \le \lim_{z\to-\infty} \overline{u}(z) = 1,$$
$$0 = \lim_{z\to-\infty} \underline{v}(z) \le \liminf_{z\to-\infty} v(z) \le \limsup_{z\to-\infty} v(z) \le \lim_{z\to-\infty} \overline{v}(z) = 0,$$

which implies that $(u(-\infty), v(-\infty)) = (1, 0)$.

We further claim that $0 < u(z) < 1$, $0 < v(z) < q(1)$ for $z \in \mathbb{R}$. For contradiction, we assume that $u(z_0) = 0$ for some $z_0 \in \mathbb{R}$, then $u'(z_0) = 0$ owing to $u(z) \geq 0$ over $\mathbb{R}$. The uniqueness of solution yields $u(z) \equiv 0$, which contradicts $u(z) \geq \underline{u}(z) > 0$ for $z < z_1$. Hence, we have $u(z) > 0$ over $\mathbb{R}$. Similar reasons lead to $v(z) > 0$ over $\mathbb{R}$. In order to prove $u(z) < 1$ over $\mathbb{R}$, we also assume that $u(z_0) = 1$ for some $z_0 \in \mathbb{R}$. It follows that $u'(z_0) = 0$ and $u''(z_0) \leq 0$. Using $u(z)$-equation of system (1.3) and the assumption (H1) and (H2), we have $0 = u''(z_0) - f(u(z_0))v(z_0) \leq -f(u(z_0))v(z_0) < 0$. Hence, this contradiction yields $u(z) < 1$ over $\mathbb{R}$. Similarly, we also get $v(z) < q(1)$ over $\mathbb{R}$ by using $u(z) < 1$ over $\mathbb{R}$.

Finally, we show that $(u', v')(-\infty) = (0, 0)$. Indeed, from $u(z)$-equation of system (1.3), we utilize the variation constants formula to deduce that

$$u'(z) = e^{c(z-\varsigma)}u'(\varsigma) - e^{cz} \int_z^{\varsigma} e^{-c\tau}[f(u(\tau))(v(\tau) - p(u(\tau)))]d\tau,$$

thus, it is easy to check that for $z \leq \varsigma$

$$c|u'(z)| \leq ce^{c(z-\varsigma)}|u'(\varsigma)| + \max_{\tau \leq \varsigma} |f(u(\tau))(v(\tau) - p(u(\tau)))|.$$

Consequently, for fixed ς, we have

$$\limsup_{z \to -\infty} |u'(z)| \leq \frac{1}{c} \left(\max_{\tau \leq \varsigma} |f(u(\tau))(v(\tau) - p(u(\tau)))| \right).$$

It follows from (H1) and (H2) that

$$f(u(-\infty))(p(u(-\infty)) - v(-\infty)) = 0,$$

then we have $w(-\infty) = 0$ by arbitrary of ς. Similar reasons give $v'(-\infty) = 0$. □

In order to obtain more details on semi-traveling waves, it is essential for us to study the asymptotic behavior of semi-traveling waves with $c \geq c^*$ at $z = -\infty$. Hence, we set $w(z) = u'(z)$, $y(z) = v'(z)$ and rewrite system (1.3) as a system of first-order ordinary differential equations in $\mathbb{R}^4$

$$\begin{cases} u'(z) = w(z), \\ w'(z) = cw(z) - f(u(z))(p(u(z)) - v(z)), \\ v'(z) = y(z), \\ dy'(z) = cy(z) - sv(z)\left(1 - \dfrac{v(z)}{q(u(z))}\right). \end{cases} \tag{3.4}$$

The eigenvalues of the linearization of (3.4) at $\mathbf{e}_0 = (1, 0, 0, 0)$ are

$$\lambda_1 = \frac{1}{2d}(c - \sqrt{c^2 - 4ds}), \quad \lambda_3 = \frac{1}{2}(c + \sqrt{c^2 - 4f(1)p'(1)}),$$

$$\lambda_2 = \frac{1}{2d}(c + \sqrt{c^2 - 4ds}), \quad \lambda_4 = \frac{1}{2}(c - \sqrt{c^2 - 4f(1)p'(1)}).$$

Obviously, for $c \geq c^*$, we have $\lambda_4 < 0 < \lambda_1, \lambda_2, \lambda_3$ due to $p'(1) < 0 < f(1)$.

Lemma 3.3. *For $c \geq c^*$, the positive solution $(u(z), v(z))$ obtained by Theorem* 3.2 *satisfies*

$$\lim_{z\to-\infty} \frac{y(z)}{v(z)} = \lambda_1 \quad \textit{and} \quad \lim_{z\to-\infty} \frac{w(z)}{u(z)-1} = \Lambda \in \{\lambda_1, \lambda_3\}.$$

Proof. Based on the choice of the upper solution and the lower solution, from Lemma 2.2, we have

$$v(z) = \begin{cases} O(e^{\lambda_1 z}), & c > c^*, \\ O(-ze^{\lambda_1 z}), & c = c^* \end{cases}$$

as $z \to -\infty$, which implies that

$$\lim_{z\to-\infty} \frac{y(z)}{v(z)} = \lambda_1.$$

Now, we show the second equality, which is quite complicated and tedious. We only consider the case $c > c^*$, since when $c = c^*$ and $\lambda_1 = \lambda_2 < (>, =)\lambda_3$, it can be similarly treated by using generalized eigenvectors and the unstable manifold theorem.

(1) If $\lambda_1, \lambda_2 \neq \lambda_3$, then the corresponding eigenvectors are given by

$$\mathbf{e}_1 = \begin{pmatrix} -1 \\ -\lambda_1 \\ -\psi(\lambda_1) \\ -\lambda_1\psi(\lambda_1) \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} -1 \\ -\lambda_2 \\ -\psi(\lambda_2) \\ -\lambda_2\psi(\lambda_2) \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} -1 \\ -\lambda_3 \\ 0 \\ 0 \end{pmatrix},$$

where

$$\psi(\lambda) = \frac{\lambda^2 - c\lambda + f(1)p'(1)}{f(1)}.$$

Thus, every solution $W(z)$ of the corresponding linearized system, which converges to $\mathbf{e}_0$ as $z \to -\infty$, is given by

$$W(z) = \mathbf{e}_0 + C_1\mathbf{e}_1 e^{\lambda_1 z} + C_2\mathbf{e}_2 e^{\lambda_2 z} + C_3\mathbf{e}_3 e^{\lambda_3 z}$$

for some constants C_i, $i = 1, 2, 3$. Applying the unstable manifold theorem yields that as $z \to -\infty$, there are α, β and γ such that

$$\begin{aligned} u(z) &= 1 - \alpha e^{\lambda_1 z} - \beta e^{\lambda_2 z} - \gamma e^{\lambda_3 z} + \text{h.o.t.}, \\ w(z) &= 0 - \alpha\lambda_1 e^{\lambda_1 z} - \beta\lambda_2 e^{\lambda_2 z} - \gamma\lambda_3 e^{\lambda_3 z} + \text{h.o.t.}, \\ v(z) &= 0 - \alpha\psi(\lambda_1)e^{\lambda_1 z} - \beta\psi(\lambda_2)e^{\lambda_2 z} + \text{h.o.t.} \end{aligned}$$

Note that $v(z) = O(e^{\lambda_1 z})$ as $z \to -\infty$, we have $\alpha \neq 0$. Then the following hold:

(a) If $\lambda_1 < \lambda_3$, then $\psi(\lambda_1) < 0$. Hence, $\alpha > 0$ due to $v(z) > 0$ for $z \in \mathbb{R}$.
(b) If $\lambda_1 > \lambda_3$, then $\psi(\lambda_1) > 0$. Hence, $\alpha < 0 < \gamma$ due to $v(z) > 0$ and $u(z) < 1$ for $z \in \mathbb{R}$.

Consequently, one can easily verify that

$$\lim_{z\to-\infty}\frac{w(z)}{u(z)-1}=\min\{\lambda_1,\lambda_3\}.$$

(2) If $\lambda_1=\lambda_3<\lambda_2$, we have a generalized eigenvector

$$\mathbf{e}_4=\begin{pmatrix}0\\ -f(1)/(2\lambda_1-c)\\ -1\\ -\lambda_1\end{pmatrix}.$$

Thus, every solution $W(z)$ of the corresponding linearized system, which converges to $\mathbf{e}_0$ as $z\to-\infty$, is given by

$$W(z)=\mathbf{e}_0+C_1(\mathbf{e}_1z+\mathbf{e}_4)e^{\lambda_1z}+C_2\mathbf{e}_2e^{\lambda_2z}$$

for some constants C_i, $i=1,2$. Applying the unstable manifold theorem yields that as $z\to-\infty$, there are α and β such that

$$\begin{aligned}u(z)&=1-\alpha ze^{\lambda_1z}-\beta e^{\lambda_2z}+\text{h.o.t.},\\ w(z)&=0-\alpha[\lambda_1z+f(1)/(2\lambda_1-c)]e^{\lambda_1z}-\beta\lambda_2e^{\lambda_2z}+\text{h.o.t.},\\ v(z)&=0-\alpha e^{\lambda_1z}-\beta\psi(\lambda_2)e^{\lambda_2z}+\text{h.o.t.}\end{aligned}$$

Note that $v(z)=O(e^{\lambda_1z})$ as $z\to-\infty$ and $v(z)>0$, we must have $\alpha<0$. Moreover, one can easily verify that

$$\lim_{z\to-\infty}\frac{w(z)}{u(z)-1}=\lambda_1. \tag{3.5}$$

(3) If $\lambda_1<\lambda_2=\lambda_3$, we have a generalized eigenvector

$$\mathbf{e}_5=\begin{pmatrix}0\\ -f(1)/(2\lambda_2-c)\\ -1\\ -\lambda_2\end{pmatrix}.$$

Thus, every solution $W(z)$ of the corresponding linearized system, which converges to $\mathbf{e}_0$ as $z\to-\infty$, is given by

$$W(z)=\mathbf{e}_0+C_1\mathbf{e}_1e^{\lambda_1z}+C_2(\mathbf{e}_2z+\mathbf{e}_5)e^{\lambda_2z}$$

for some constants C_i, $i=1,2$. Applying the unstable manifold theorem yields that as $z\to-\infty$, there are α and β such that

$$\begin{aligned}u(z)&=1-\alpha e^{\lambda_1z}-\beta ze^{\lambda_2z}+\text{h.o.t.},\\ w(z)&=0-\alpha\lambda_1e^{\lambda_1z}-\beta[\lambda_2z+f(1)/(2\lambda_2-c)]e^{\lambda_2z}+\text{h.o.t.},\\ v(z)&=0-\alpha\psi(\lambda_1)e^{\lambda_1z}-\beta e^{\lambda_2z}+\text{h.o.t.}\end{aligned}$$

Similarly, we have $\alpha>0$ and (3.5) still hold. Therefore, we complete the proof. □

Remark 3.4. From Theorem 3.2, we know that $v(z) > 0$ and $u(z) < 1$ over $\mathbb{R}$. Combining with Lemma 3.3, one can infer that $y(z) > 0 > w(z)$ for $z \ll -1$, which will be used to prove the monotonicity of traveling waves connecting $(1, 0)$ and $(0, \mu)$ in Theorem 4.5.

Finally, let us end this section by proving Theorem 1.1.

Proof of Theorem 1.1. We investigate the linearization of $v(z)$-equation of system (1.3) at $(1, 0)$, which provides two eigenvalues λ_1 and λ_2 defined in (3.1). If $|c| < c^*$, then λ_1 and λ_2 are a pair of complex eigenvalues, implying that $v(z)$ changes the signs many times as $z \to -\infty$, thus, we arrive at a contradiction with the positivity of $v(z)$. On the other hand, if $c < -c^*$, then $\lambda_1 < \lambda_2 < 0$ and $v(z)$ is unbounded as $z \to -\infty$, which contradicts $0 < v(z) < q(1)$ over $\mathbb{R}$. Since neither is feasible, system (1.2) has no semi-traveling wave if $c < c^*$, which concludes the result by coupling Theorem 3.2. □

4. The Existence of Traveling Waves

In this section, our main purpose is to prove the existence of traveling waves of system (1.2). First, inspired by [18, 24, 36, 40], we shall develop two derivative estimates to produce an invariant set of system (3.4), which is a required and crucial effort. Then semi-traveling waves obtained in Theorem 3.2 can be viewed as good candidates for traveling waves. Finally, we deduce the right-hand tail limit of such solutions, which establishes the existence of traveling waves. In the sequence, we always assume that $c \geq c^*$.

We start by recalling LaSalle's invariance principle.

Proposition 4.1 ([21]). *Consider the following initial value problem*

$$\begin{cases} y' = f(y), \\ y(0) = y_0, \end{cases} \tag{4.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies Lipschitz condition. Let $\Sigma \subseteq \mathbb{R}^n$ be an open set. Suppose $y(t, y_0)$ is a solution of Eq. (4.1) *which is positively invariant in Σ. If there is a continuous and bounded below function $V(y) : \Sigma \to \mathbb{R}^n$ such that the orbital derivative of $V(y)$ along $y(t, y_0)$ is non-positive, i.e.*

$$\frac{dV}{dt} = \text{grad } V(y) \cdot f(y) \leq 0,$$

then the ω-limit set of $y(t, y_0)$ is contained in the largest invariant set

$$I = \{y \in \Sigma : \text{grad } V(y) \cdot f(y) = 0\}.$$

4.1. *Derivative estimates*

In this section, we shall develop two derivative estimates of $u(z)$ and $v(z)$.

Lemma 4.2. *There is a large enough $K > q(1)/c$ such that for $z \in \mathbb{R}$,*

$$-Kf(u(z)) < w(z) < Kf(u(z)).$$

Proof. To prove the above inequalities, we define functions

$$\phi_1(z) = w(z) - Kf(u(z)),$$

$$\phi_2(z) = w(z) + Kf(u(z)).$$

It is sufficient to show that $\phi_1(z) < 0$ and $\phi_2(z) > 0$ over $\mathbb{R}$. By recalling Theorem 3.2, we have $0 < u(z) < 1$ and $(u(-\infty), w(-\infty)) = (1, 0)$. Thus, there are $K > 0$ and $z_0 \ll -1$ such that

$$\phi_1(z) < 0 \quad \text{and} \quad \phi_2(z) > 0 \quad \text{for } z \le z_0.$$

(1) We show that $\phi_1(z) < 0$ for $z > z_0$ by contradiction. Assume that there is a $z_1 > z_0$ such that $\phi_1(z_1) = 0$ and $\phi_1'(z_1) \ge 0$. Since $u(z)$ and $v(z)$ are bounded, and $w(z_1) = Kf(u(z_1))$, then for large enough $K > 0$

$$\begin{aligned} 0 \le \phi_1'(z_1) &= w'(z_1) - Kf'(u(z_1))w(z_1) \\ &= cw(z_1) - f(u(z_1))p(u(z_1)) + f(u(z_1))v(z_1) - Kf'(u(z_1))w(z_1) \\ &= [cK - p(u(z_1)) + v(z_1) - K^2 f'(u(z_1))]f(u(z_1)) < 0. \end{aligned}$$

Hence, the contradiction yields $\phi_1(z) < 0$ for $z > z_0$.

(2) We show that $\phi_2(z) > 0$ for $z > z_0$ by contradiction. Assume that there is a $z_1 > z_0$ such that $\phi_2(z_1) = 0$ and $\phi_2'(z_1) \le 0$. We claim that there is a $z_2 > z_1$ such that $\phi_2(z_2) = 0$ and $\phi_2'(z_2) \ge 0$. To see this, we further assume that $\phi_2(z) < 0$ for all $z > z_1$, that is, $w(z) < -Kf(u(z))$ for all $z > z_1$. Note that $0 < u(z) < 1$, $0 < v(z) < q(1)$ over $\mathbb{R}$ and $p(u) > 0$ for $u \in (0, 1)$, then for $z > z_1$

$$\begin{aligned} w'(z) &= cw(z) + f(u(z))v(z) - f(u(z))p(u(z)) \\ &< [-cK + v(z) - p(u(z))]f(u(z)) \\ &< (q(1) - cK)f(u(z)) < 0 \end{aligned}$$

as long as $K > q(1)/c$. Thus, it is easy to check that for $z > z_1$

$$w(z) < w(z_1) = -Kf(u(z_1)) < 0,$$

which contradicts the boundedness of $u(z)$. Hence, the claim is valid. On the other hand, at $z = z_2$, $w(z_2) = -Kf(u(z_2))$, we also get for large enough $K > 0$

$$\begin{aligned} 0 \le \phi_2'(z_2) &= w'(z_2) + Kf'(u(z_2))w(z_2) \\ &= cw(z_2) - f(u(z_2))p(u(z_2)) + f(u(z_2))v(z_2) + Kf'(u(z_2))w(z_2) \\ &= [-(K^2 f'(u(z_2)) + cK) + v(z_2) - p(u(z_2))]f(u(z_2)) < 0 \end{aligned}$$

due to the boundedness of $u(z)$ and $v(z)$. Hence, this contradiction yields $\phi_2(z) > 0$ for $z > z_0$. ◻

Lemma 4.3. *If* $c \geq c^* = 2\sqrt{ds}$, *then for* $z \in \mathbb{R}$,

$$-\frac{sq(1)}{cq(0)}v(z) \leq y(z) \leq \frac{c}{2d}v(z).$$

Proof. To prove the above inequalities, we define functions

$$\psi_1(z) = y(z) - \frac{c}{2d}v(z),$$

$$\psi_2(z) = y(z) + \frac{sq(1)}{cq(0)}v(z).$$

It is sufficient to show that $\psi_1(z) \leq 0$ and $\psi_2(z) \geq 0$ over $\mathbb{R}$.

(1) We show that $\psi_1(z) \leq 0$ over $\mathbb{R}$ by contradiction. Assume that there is a z_0 such that $\psi_1(z_0) > 0$, then $v(z)$-equation of system (1.3) gives for $c \geq c^* = 2\sqrt{ds}$

$$\begin{aligned} d\psi_1'(z) - \frac{c}{2}\psi_1(z) &= dy'(z) - cy(z) + \frac{c^2}{4d}v(z) \\ &= \left(\frac{sv(z)}{q(u(z))} + \frac{c^2}{4d} - s\right)v(z) \\ &\geq \frac{sv^2(z)}{q(u(z))} > 0. \end{aligned}$$

Hence, the comparison principle yields $\psi_1(z) > e^{c(z-z_0)/2d}\psi_1(z_0) > 0$ for $z > z_0$. Using the expression of $\psi_1(z)$ and the comparison principle again, we obtain

$$v(z) > e^{c(z-z_0)/2d}v(z_0) \quad \text{for } z > z_0.$$

Therefore, we have $v(z) \to +\infty$ as $z \to +\infty$, which contradicts the boundedness of $v(z)$.

(2) We show that $\psi_2(z) \geq 0$ over $\mathbb{R}$. From Lemma 3.3, we have for $c \geq c^*$

$$\lim_{z\to-\infty} \frac{y(z)}{v(z)} = \lambda_1 > 0,$$

then there is a $z_0 \ll -1$ such that for $z \leq z_0$

$$\psi_2(z) = v(z)\left(\frac{y(z)}{v(z)} + \frac{sq(1)}{cq(0)}\right) > 0.$$

For contradiction, we assume that there is a $z_1 > z_0$ such that $\psi_2(z_1) = 0$ and $\psi_2'(z_1) \leq 0$. Similar to the proof of Lemma 4.2, $\psi_2(z_2) = 0$ and $\psi_2'(z_2) \geq 0$ for some $z_2 > z_1$. Actually, since $0 < u(z) < 1$, $0 < v(z) < q(1)$ over $\mathbb{R}$ and $q(u(z)) > q(0)$, then as long as $y'(z) \leq -sq(1)v(z)/cq(0)$, we have

$$dy'(z) = cy(z) - sv(z)\left(1 - \frac{v(z)}{q(u(z))}\right) \leq sv(z)\left(\frac{v(z)}{q(u(z))} - \frac{q(1)}{q(0)} - 1\right) < 0. \tag{4.2}$$

Therefore, if $\psi_2(z) < 0$ for all $z > z_1$, that is, for $z > z_1$

$$y(z) < -\frac{sq(1)}{cq(0)}v(z),$$

then (4.2) yields for $z > z_1$,

$$y(z) < y(z_1) = -\frac{sq(1)}{cq(0)}v(z_1) < 0,$$

which contradicts the positivity of $v(z)$. Thanks to $\psi_2'(z_2) \geq 0$ and $y(z_2) = -sq(1)v(z_2)/cq(0)$, we have

$$y'(z_2) \geq -\frac{sq(1)}{cq(0)}y(z_2) = \left(\frac{sq(1)}{cq(0)}\right)^2 v(z_2) > 0.$$

Thus, from $0 < v(z_2) < q(1)$ and $q(u(z_2)) > q(0)$, we deduce that

$$0 = dy'(z_2) - cy(z_2) + sv(z_2)\left(1 - \frac{v(z_2)}{q(u(z_2))}\right) \geq sv(z_2)\left(\frac{q(1)}{q(0)} - \frac{v(z_2)}{q(u(z_2))} + 1\right) > 0,$$

which leads to a contradiction. Hence, we complete the proof. □

Then according to Lemmas 4.2 and 4.3, the set

$$\Sigma = \left\{\chi(z) | 0 < u < 1, 0 < v < q(1), -Kf(u) < w < Kf(u), -\frac{2sq(1)}{cq(0)}v < y < \frac{c}{d}v\right\}$$

is an invariant set of system (3.4), where $\chi(z) := (u, w, v, y)(z)$.

4.2. *Convergence of semi-traveling waves*

In this section, we will derive the right-hand tail limit of semi-traveling waves obtained in Theorem 3.2. Specifically, based on the above open set Σ, the appropriate Lyapunov functions are constructed such that the candidates are traveling waves.

We first prove Theorem 1.2. For convenience, let us recall it again.

Theorem 4.4. *Assume that* (H1) *and* (H2) *hold and* $h(u) = 1$. *For* $c \geq c^*$, *if* $p(0) > \mu$ *and* $(u^* + \mu)f(u) - (u - u^*)(u + \mu)f'(u) > 0$ *for* $u \in (0, 1)$, *then system* (1.2) *has a traveling wave connecting* $(1, 0)$ *and* (u^*, v^*).

Proof. Let

$$H(u,v) = \int_{u^*}^{u} \frac{\eta - u^*}{q(\eta)f(\eta)} d\eta + \frac{1}{s}\int_{v^*}^{v} \frac{\eta - v^*}{\eta} d\eta.$$

It is easy to check that $H(u,v) \geq 0$ for $(u,v) \in (0,1) \times (0,q(1))$. In fact, due to $H(u^*,v^*) = 0$, we have for $(u,v) \in (0,1) \times (0,q(1))$

$$\begin{aligned} H(u,v) &= H_u(\hat{u},\hat{v})(u-u^*) + H_v(\hat{u},\hat{v})(v-v^*) \\ &= \frac{\hat{u}-u^*}{q(\hat{u})f(\hat{u})}(u-u^*) + \frac{\hat{v}-v^*}{s\hat{v}}(v-v^*) \\ &= \theta\left(\frac{(u-u^*)^2}{q(\hat{u})f(\hat{u})} + \frac{(v-v^*)^2}{s\hat{v}}\right) \geq 0, \end{aligned}$$

where $\hat{u} = u^* + \theta(u-u^*) > 0$ and $\hat{v} = v^* + \theta(v-v^*) > 0$ with $\theta \in (0,1)$. Then we define Lyapunov function $\mathcal{L} : \mathbb{R}^4 \to \mathbb{R}$ by

$$\mathcal{L}(u,w,v,y) = cH(u,v) - w\partial_u H - dy\partial_v H,$$

thus, $\mathcal{L}(u,w,v,y)$ is continuous function with a lower bound since for $(u,w,v,y) \in D$

$$w\partial_u H = \frac{w(u-u^*)}{f(u)q(u)} \leq \frac{w(u+u^*)}{q(0)f(u)} \leq \frac{2w}{q(0)f(u)} < \frac{2K}{q(0)},$$

$$y\partial_v H = \frac{y(v-v^*)}{sv} \leq \frac{1}{s}\left(\frac{c}{d}v - \frac{y}{v}v^*\right) \leq \frac{q(1)}{s}\left(\frac{c}{d} + \frac{2sv^*}{cq(0)}\right).$$

From (3.4), the orbital derivative of $\mathcal{L}$ along $\chi(z)$ is

$$\begin{aligned} \mathcal{L}'(\chi(z)) &= (cw - w')\partial_u H + (cy - dy')\partial_v H - w^2\partial_u^2 H - dy^2\partial_v^2 H \\ &= f(u)(p(u)-v)\partial_u H + sv(1 - v/q(u))\partial_v H - w^2\partial_u^2 H - dy^2\partial_v^2 H \\ &= [(u-u^*)(p(u)-v) + (v-v^*)(q(u)-v)]/q(u) - w^2\partial_u^2 H - dy^2\partial_v^2 H. \end{aligned}$$

Using the expression of $p(u^*) = q(u^*) = v^*$, we further obtain

$$(u-u^*)(p(u)-v) = (u-u^*)(p(u)-p(u^*)) + (u-u^*)(v^*-v),$$

$$(v-v^*)(q(u)-v) = (v-v^*)(q(u)-q(u^*)) - (v-v^*)^2.$$

Substituting the above identities into $\mathcal{L}'(\chi(z))$, we get

$$\begin{aligned} \mathcal{L}'(\chi(z)) = {} & [(u-u^*)(p(u)-p(u^*)) - (v-v^*)^2]/q(u) - w^2\partial_u^2 H \\ & - dy^2\partial_v^2 H + G(u,v), \end{aligned}$$

where

$$\begin{aligned} G(u,v) &= (v-v^*)[(q(u)-q(u^*)) - (u-u^*)]/q(u) \\ &= (v-v^*)[(u-u^*) - (u-u^*)]/q(u) = 0 \end{aligned}$$

due to $q(u) = uh(u) + \mu = u + \mu$. Moreover, a simple calculation gives

$$\partial_v^2 H = \frac{v^*}{sv^2} \quad \text{and} \quad \partial_u^2 H = \frac{(u^* + \mu)f(u) - (u - u^*)(u + \mu)f'(u)}{(u + \mu)^2 f^2(u)}.$$

Thus, (H2) and (P) imply that the first and third terms are non-positive in $\mathcal{L}'(\chi(z))$. Hence, the orbital derivative of $\mathcal{L}$ along $\chi(z)$ is non-positive, and it is zero if and only if $\chi(z) = (u^*, 0, v^*, 0)$. On the other hand, $p(0) > \mu$ implies that there is a unique positive equilibrium (u^*, v^*) in Σ. Therefore, we complete the proof by using LaSalle's invariance principle. □

Finally, we show Theorem 1.3. By recalling that

$$\mathcal{A} = \{v \in C^2(\mathbb{R}) \,|\, 0 < v(z) < \mu \text{ and } y(z) > 0 \text{ over } \mathbb{R}\},$$

$$\mathcal{B} = \left\{ v \in C^2(\mathbb{R}) \middle| \begin{array}{l} \exists z_v \text{ such that } v(z_v) = \mu, y(z) > 0 \text{ in } (-\infty, z_v) \\ \text{and } v(z) > \mu \text{ in } (z_v, +\infty) \end{array} \right\},$$

then Theorem 1.3 is repeated as below.

Theorem 4.5. *Assume that* (H1)–(H3) *hold. For* $c \geq c^*$, *if*

$$f(u) = ug(u) \quad \textit{and} \quad p(u) = (1 - u)/g(u),$$

where $g(u)$ *satisfies*

$$\min_{u \in [0,1]} g(u) > \frac{h(1)(h(1) + \mu)}{\mu^2 h(0)}, \tag{4.3}$$

then system (1.2) *has a traveling wave* $(u(z), v(z))$ *connecting* $(1, 0)$ *and* $(0, \mu)$, *where* $v(z) \in \mathcal{A} \cup \mathcal{B}$ *and* $u'(z) < 0$ *over* $\mathbb{R}$.

Proof. First, we show the existence by LaSalle's invariance principle. It follows from (4.3) that there is a ϱ satisfying

$$\frac{sh(1)}{\mu(\min_{u \in [0,1]} g(u))} < \varrho < \frac{s\mu h(0)}{h(1) + \mu}. \tag{4.4}$$

Then we define Lyapunov function $\mathcal{L} : \mathbb{R}^4 \to \mathbb{R}$ by

$$\mathcal{L}(\chi(z)) = \varrho(cu - w) + c\int_{q(0)}^{v} \frac{\eta - q(0)}{\eta} d\eta - dy + \frac{dq(0)y}{v}.$$

Since $q(u) - q(0) = q(u) - \mu = uh(u)$ and (3.4), after a direct calculation, we get

$$\begin{aligned} \mathcal{L}'(\chi(z)) &= \varrho(cw - w') + (1 - q(0)/v)(cy - dy') - dq(0)(y/v)^2 \\ &= \varrho u(1 - u - g(u)v) + s(v - q(0))(1 - v/q(u)) - dq(0)(y/v)^2 \\ &= -\varrho u^2 + \varrho u(1 - g(u)v) - s(v - q(0))^2/q(u) \\ &\quad + s(v - q(0))(q(u) - q(0))/q(u) - dq(0)(y/v)^2 \end{aligned}$$

$$= -\varrho u^2 - s(v - q(0))^2/q(u) - dq(0)(y/v)^2$$
$$+ \varrho u[1 - g(u)v + s(v - q(0))h(u)/\varrho q(u)]$$
$$= -\varrho u^2 - s(v - q(0))^2/q(u) - dq(0)(y/v)^2 + \varrho uC(u),$$

where

$$C(u) = 1 - g(u)v + s(v - q(0))h(u)/\varrho q(u).$$

Through the monotonicity of $h(u)$ and $0 < u < 1$, we have

$$\begin{aligned} C(u) &= [(\varrho q(u) - sq(0)h(u)) + v(sh(u) - \varrho q(u)g(u))]/\varrho q(u) \\ &< [(\varrho q(1) - sq(0)h(0)) + v(sh(1) - \varrho q(0)g(u))]/\varrho q(u) \\ &= \{[\varrho(h(1) + \mu) - s\mu h(0)] + v(sh(1) - \varrho\mu g(u))\}/\varrho q(u). \end{aligned}$$

Automatically, $C(u) < 0$ for $u \in [0, 1]$ by the choice of ϱ. Then the orbital derivative of $\mathcal{L}$ along $\chi(z)$ is non-positive and it is zero if and only if $u = 0$, $v = \mu$, $y = 0$. Thus, LaSalle's invariance principle yields $\chi(z) \to (0, w(+\infty), \mu, 0)$ as $z \to +\infty$ if $w(+\infty)$ exists. Since $u(z)$, $v(z)$ and $w(z)$ are bounded for $z \in \mathbb{R}$, $u(z)$-equation gives that $w(z)$ is uniformly continuous. And we further infer that $w(+\infty) = 0$ by Bărbalat's Lemma[8]. Hence, the existence of traveling waves is proven.

Next, we show $v(z) \in \mathcal{A} \cup \mathcal{B}$. We claim that $y(z) \neq 0$ as long as z satisfies $0 < v(z) \leq \mu$. Actually, we assume that there is a $z_0 \in \mathbb{R}$ such that $y(z_0) = 0$ and $0 < v(z_0) \leq \mu$. Since $0 < u(z_0) < 1$ and $\mu = q(0) < q(u(z_0))$ by the monotonicity of $q(u)$, we have

$$dy'(z_0) = -sv(z_0)\left(1 - \frac{v(z_0)}{q(u(z_0))}\right) < -sv(z_0)\left(1 - \frac{\mu}{q(0)}\right) = 0,$$

implying that $v(z)$ attains a local maximum at z_0. Thus, $0 < v(z_1) < \mu$ and $y(z_1) < 0$ for some $z_1 > z_0$. Observe that

$$dy'(z) = cy(z) - sv(z)\left(1 - \frac{v(z)}{q(u(z))}\right) < 0$$

as long as z satisfies $y(z) \leq 0$ and $0 < v(z) \leq \mu$, we have $y'(z) < 0$ and $y(z) < y(z_1) < 0$ for $z > z_1$, which contradicts the positive of $v(z)$. Therefore, we prove the assertion of the claim. Since $v(-\infty) = 0$ and $v(+\infty) = \mu$, $y(z) > 0$ as long as z satisfies $0 < v(z) \leq \mu$, meanwhile, it holds that either $v(z) \in (0, \mu)$ over $\mathbb{R}$ or there exists a z_v such that $v(z_v) = \mu$. For the former case, clearly, $v(z) \in \mathcal{A}$. For the latter case, $y(z) > 0$ for $z \leq z_v$. On the other hand, there is no point $z > z_v$ with $v(z) = \mu$ and $y(z) \leq 0$, which implies that $v(z) \in \mathcal{B}$.

Finally, we show that $w(z) < 0$ over $\mathbb{R}$, which can be considered in two cases.

For the case $v \in \mathcal{A}$. For contradiction, we assume that there is $z_0 \in \mathbb{R}$ such that $w(z_0) = 0$. From Remark 3.4, we know that $w(z) < 0$ for $z \ll -1$, then we can reasonably set

$$\bar{z} = \sup\{z \in (-\infty, z_0] : w(\varsigma) < 0 \text{ for } \varsigma \in (-\infty, z)\}.$$

It follows from the definition of $\bar{z}$ that $w(\bar{z}) = 0$ and $w'(\bar{z}) \geq 0$. Thus, $u(z)$-equation of system (1.3) gives

$$v(\bar{z}) \geq p(u(\bar{z})). \tag{4.5}$$

Using $u(+\infty) = 0$ and the definition of $\bar{z}$, we can define

$$\hat{z} = \inf\{z \in (\bar{z}, +\infty) : w(\varsigma) > 0 \text{ for } \varsigma \in (\bar{z}, z)\}.$$

Obviously, $u(\hat{z}) > u(\bar{z})$, $w(\hat{z}) = 0$ and $w'(\hat{z}) \leq 0$. Then $u(z)$-equation gives

$$v(\hat{z}) \leq p(u(\hat{z})). \tag{4.6}$$

Since $p'(u) < 0$ for $u > 0$, we see that

$$v(\bar{z}) \geq p(u(\bar{z})) > p(u(\hat{z})) \geq v(\hat{z}),$$

which contradicts $y(z) > 0$ over $\mathbb{R}$. Therefore, we have $w(z) < 0$ over $\mathbb{R}$ if $v \in \mathcal{A}$.

For the another case $v \in \mathcal{B}$, we first claim that $w(z) < 0$ if $z \geq z_v$. Otherwise, we assume that $w(z_0) \geq 0$ for some $z_0 \geq z_v$. Since $v(z) \geq \mu$ for $z \geq z_v$, we deduce that for $z \geq z_v$

$$\begin{aligned} w'(z) - cw(z) &= f(u(z))(v(z) - p(u(z))) \\ &\geq f(u(z))(\mu - p(0)) > 0, \end{aligned} \tag{4.7}$$

here we have used the fact that

$$p(0) = \frac{1}{g(0)} < \frac{1}{\min\limits_{u \in [0,1]} g(u)} < \frac{\mu^2 h(0)}{h(1)(h(1) + \mu)} < \mu.$$

For the case $w(z_0) > 0$, (4.7) and the comparison principle give for all $z > z_0$

$$w(z) > w(z_0)e^{c(z - z_0)}, \tag{4.8}$$

which contradicts the boundedness of $u(z)$. For the case $w(z_0) = 0$, (4.8) yields that $w(z_1) > 0$ for some $z_1 > z_0$, using (4.7) and the comparison principle again, we obtain all $z > z_1$

$$w(z) > w(z_1)e^{c(z - z_1)},$$

which contradicts the boundedness of $u(z)$. Thus, we prove the assertion of the claim. Therefore, it is sufficient to prove that $w(z) < 0$ for $z < z_v$. Similar to the case $v(z) \in \mathcal{A}$, we assume that there is $z_0 < z_v$ such that $w(z_0) = 0$ and define

$$\bar{z} = \sup\{z \in (-\infty, z_0] : w(\varsigma) < 0 \text{ for } \varsigma \in (-\infty, z)\}.$$

Obviously, from the definition of $\bar{z}$, we have $w(\bar{z}) = 0$ and $w'(\bar{z}) \geq 0$. Then inequality (4.5) is still valid. Note that $w(z_v) < 0$ from the above claim, then there is a $\hat{z} \in (\bar{z}, z_v)$ such that $u(z)$ attains a local maximum at $\hat{z}$, and inequality (4.6) also remains valid. These again contradict the monotonicity of $v(z)$. Above all, we complete the proof. □

Remark 4.6. Suppose system (1.2) has a traveling wave $(u(z), v(z))$ connecting $(1, 0)$ and $(0, \mu)$. If $p(0) < \mu$, then $v(z) \in \mathcal{A} \cup \mathcal{B}$ and $u'(z) < 0$ over $\mathbb{R}$.

5. Applications

In this section, we apply Theorems 1.2 and 1.3 to modified Leslie–Gower system with Holling-II functional response or Lotka–Volterra type functional response, then some numerical simulations are carried out by using MATLAB. It should be pointed out that we achieve a better result on the existence of traveling waves connecting $(1, 0)$ and (u^*, v^*) by constructing a novel Lyapunov function for system with Lotka–Volterra type functional response.

5.1. *System* (1.2) *with Holling*-II *type functional response*

Consider system

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \Delta u(x,t) + u(x,t)\left(1 - u(x,t) - \dfrac{av(x,t)}{u(x,t)+e_1}\right), \\ \dfrac{\partial v(x,t)}{\partial t} = d\Delta v(x,t) + sv(x,t)\left(1 - \dfrac{v(x,t)}{u(x,t)+e_2}\right), \end{cases} \tag{5.1}$$

where a, e_1, d, s and e_2 are positive constants. Then

$$f(u) = \frac{au}{u+e_1}, \quad p(u) = \frac{1}{a}(1-u)(u+e_1), \quad q(u) = u + e_2, \quad h(u) = 1, \quad \mu = e_2.$$

For convenience, we define

$$\gamma(x) = (1-x)(x+e_1)/(x+e_2), \quad x \in (0, 1).$$

From Theorem 1.2, we have the following result on the existence of traveling waves connecting $(1, 0)$ and (u^*, v^*).

Theorem 5.1. *For $c \geq c^*$, there is a constant*

$$\bar{a} = \begin{cases} e_1/e_2 & \text{if } e_1 \in [1, +\infty) \cap (0, e_2], \\ \min\left\{e_1/e_2, \gamma\left(\dfrac{e_1 - e_2}{1 + 2e_1 + e_1e_2}\right)\right\} & \text{if } e_1 \in [1, +\infty) \cap (e_2, +\infty), \end{cases}$$

such that if $a < \bar{a}$, then system (5.1) *has a traveling wave connecting* $(1, 0)$ *and* (u^*, v^*).

Proof. Clearly, Assumption 1.1 holds when $e_1 \geq 1$. If $a < e_1/e_2$, there is a $u^* \in (0, 1)$ satisfying $K(u) = u^2 - (1 - a - e_1)u + (ae_2 - e_1) = 0$, such that system has a unique positive equilibrium (u^*, v^*). Then we only check $(u^* + \mu)f(u) - (u - u^*)(u + \mu)f'(u) > 0$ for $u \in (0, 1)$, that is, $B(u) = (u^* + e_2 - e_1)u^2 + 2u^*e_1u + e_1e_2u^* > 0$ for $u \in (0, 1)$.

(1) When $u^* = e_1 - e_2 \in (0,1)$, that is, $e_1 \in (e_2, e_2+1)$ and $a = \gamma(e_1 - e_2)$. It is obvious that $B(u) > 0$ for $u \in (0,1)$.

(2) When $u^* \in (e_1 - e_2, 1)$, we have $B'(u) > 0$ for $u \in (0,1)$ and $B(u) > B(0) = e_1 e_2 u^* > 0$.

   (a) If $e_1 \in (0, e_2]$, then $u^* \in (0,1) \subseteq (e_1 - e_2, 1)$.

   (b) If $e_1 \in (e_2, e_2+1)$, then $u^* \in (e_1 - e_2, 1)$ is equivalent to $K(e_1 - e_2) < 0$ from the graph of $K(u)$, that is, $a < \gamma(e_1 - e_2)$.

(3) When $u^* \in (0, e_1 - e_2)$, $B(u)$ is a concave function. Due to $B(0) > 0$, then it is sufficient to prove that $B(1) \geq 0$, that is

$$e_1 - e_2 > u^* \geq \Lambda := \frac{e_1 - e_2}{1 + 2e_1 + e_1 e_2}.$$

   (a) If $e_1 \in (e_2, e_2+1)$, then $u^* \in [\Lambda, e_2 - e_1)$, which is equivalent to $K(\Lambda) \leq 0$ and $K(e_1 - e_2) > 0$ from the graph of $K(u)$. Note that $\gamma(x)$ is decreasing on x if $e_1 > e_2$, thus, we have $\gamma(e_1 - e_2) < a \leq \gamma(\Lambda)$.

   (b) If $e_1 \in [e_2 + 1, +\infty)$, then $u^* \in [\Lambda, 1)$, that is, $a \leq \gamma(\Lambda)$.

Based on (1), (2)(b) and (3), we know that $B(u) > 0$ for $u \in (0,1)$ if

$$a < \min\left\{e_1/e_2, \gamma\left(\frac{e_1 - e_2}{1 + 2e_1 + e_1 e_2}\right)\right\} \quad \text{for } e_1 \in (e_2, +\infty).$$

To summarise, we complete the proof. □

Next, according to Theorem 1.3, we have the following result.

Theorem 5.2. *Assume that $c \geq c^*$, if $e_1 \geq 1$ and $ae_2^2 > (1+e_1)(1+e_2)$, then system* (5.1) *has a traveling wave connecting* $(1,0)$ *and* $(0, e_2)$ *and satisfying $v(z) \in \mathcal{A} \cup \mathcal{B}$ and $u'(z) < 0$ over $\mathbb{R}$.*

Proof. Clearly, if $e_1 \geq 1$, then Assumption 1.1 holds, and $g(u) = a/(u + e_1)$. From Theorem 1.3, we complete the proof. □

In the following, we present some numerical simulations using MATLAB. Consider the initial value

$$u(x,0) = 1, \quad -200 \leq x \leq 200 \quad \text{and} \quad v(x,0) = \begin{cases} 0 & \text{for } -200 < x \leq 100, \\ 0.1 & \text{for } 100 < x \leq 200. \end{cases} \tag{5.2}$$

First of all, let $d = 1$, $s = 0.5$, $e_1 = 2$ and $e_2 = 1.2$, then $\bar{a} \approx 1.4373$. At this time, we set $a = 1.4$, then $(u^*, v^*) \approx (0.1266, 1.3266)$. On the other hand, let $d = 1$, $s = 0.5$, $e_1 = 1.2$ and $e_2 = 1.4$, then $\bar{a} \approx 0.8571$. We further choose $a = 0.7$, then $(u^*, v^*) = (0.2, 1.6)$. Therefore, the above two sets of parameters satisfy different conditions in Theorem 5.1. Under the initial value (5.2), one can observe a traveling wave connecting $(1, 0)$ and (u^*, v^*) of system (5.1) from Figs. 1 and 2. Moreover, we also select $d = 1$, $s = 0.5$, $e_1 = 1.2$, $e_2 = 0.5$ and $a = 15$, which satisfy the conditions in Theorem 5.2, then system (5.1) has a traveling wave connecting $(1, 0)$ and $(0, e_2)$ from Fig. 3.

5.2. *System* (1.2) *with Lotka–Volterra type functional response*

Consider system

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \Delta u(x,t) + u(x,t)(1 - u(x,t) - av(x,t)), \\ \dfrac{\partial v(x,t)}{\partial t} = d\Delta v(x,t) + sv(x,t)\left(1 - \dfrac{v(x,t)}{u(x,t)+e_2}\right), \end{cases} \tag{5.3}$$

where a, d, s and e_2 are positive constants. Then

$$f(u) = au, \quad p(u) = \frac{1-u}{a}, \quad q(u) = u + e_2, \quad h(u) = 1, \quad \mu = e_2$$

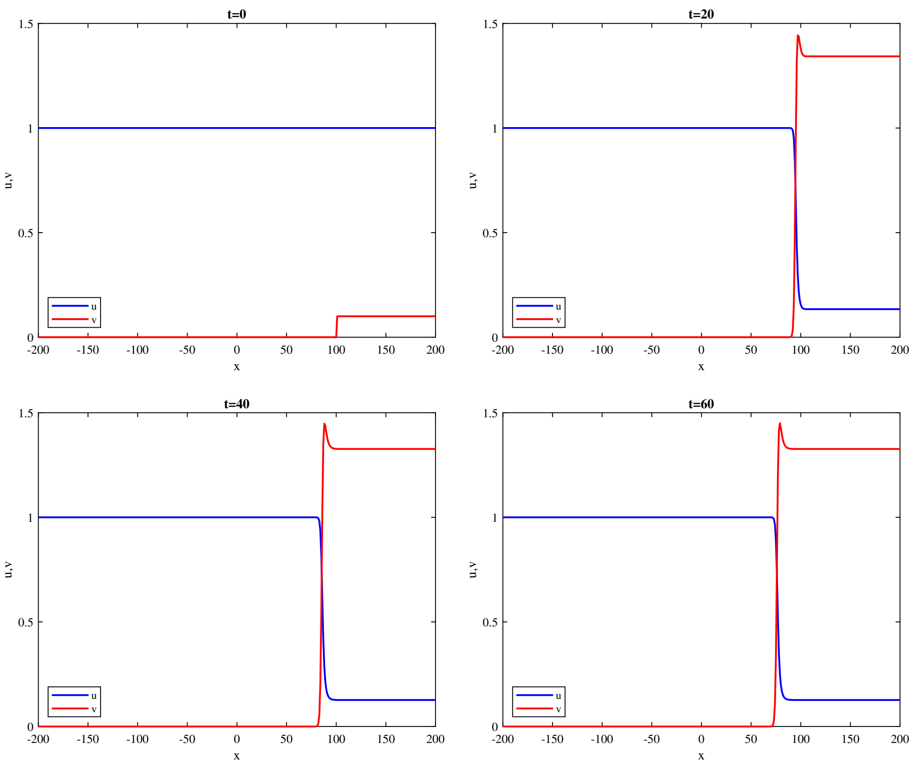

Fig. 1. Traveling wave of system (5.1) at different times using the parameters $d = 1$, $s = 0.5$, $e_1 = 2$, $e_2 = 1.2$ and $a = 1.4$.

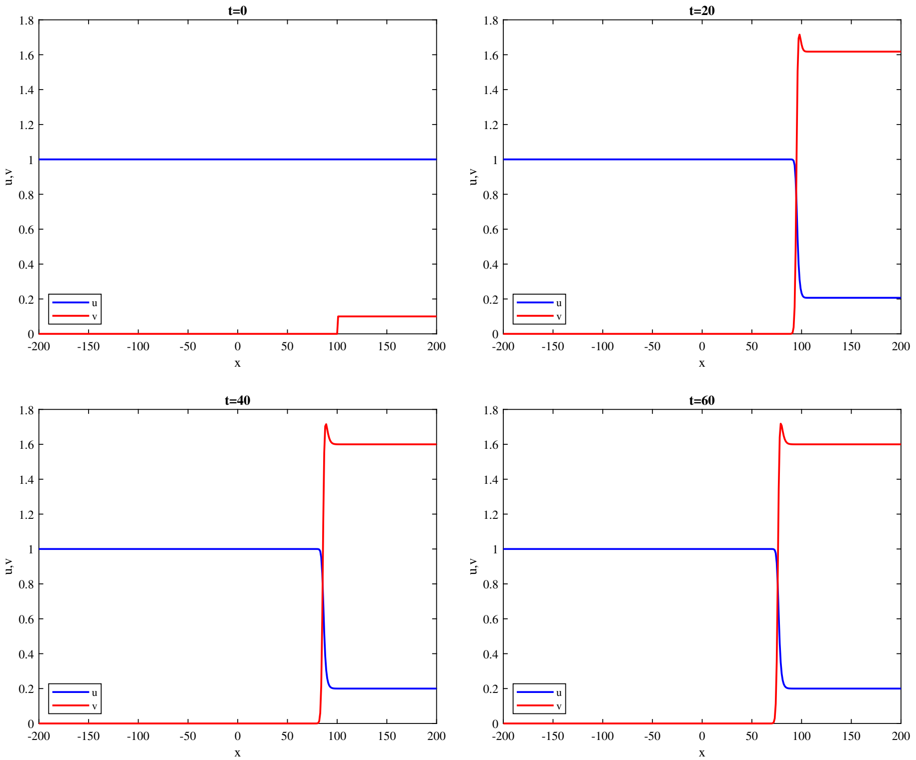

Fig. 2. Traveling wave of system (5.1) at different times using the parameters $d = 1$, $s = 0.5$, $e_1 = 1.2$, $e_2 = 1.4$ and $a = 0.7$.

and

$$(u^*, v^*) = \left(\frac{1 - ae_2}{1 + a}, \frac{1 + e_2}{1 + a}\right).$$

From Theorem 1.2, one can easily verify that if $a \leq 1/(1 + e_2)$, then system (5.3) has a traveling wave connecting $(1, 0)$ and (u^*, v^*). However, inspired by [16, 38], we obtain a better result by constructing a novel Lyapunov function.

We need the following lemma to proceed.

Lemma 5.3. *The polynomial* $S(a) = a^3 + a^2 - 16a(1 + e_2) - 16(1 + e_2)(2 + e_2)$ *has a unique positive root* $\tilde{a}$ *satisfying* $\tilde{a} > 1/(1 + e_2)$.

Proof. Note that the coefficient sign of $S(a)$ changes only once, it follows from Descartes' Rule of Signs that $S(a)$ has a unique positive root $\tilde{a}$. To show that $\tilde{a} > 1/(1 + e_2)$, we only need to prove $S(1/(1 + e_2)) < 0$. Indeed, we have

$$-(1+e_2)^3 S\left(\frac{1}{1+e_2}\right) = 16(1+e_2)^5 + 16(1+e_2)^4 + 16(1+e_2)^3 - (1+e_2) - 1 > 0.$$

Therefore, we complete the proof. □

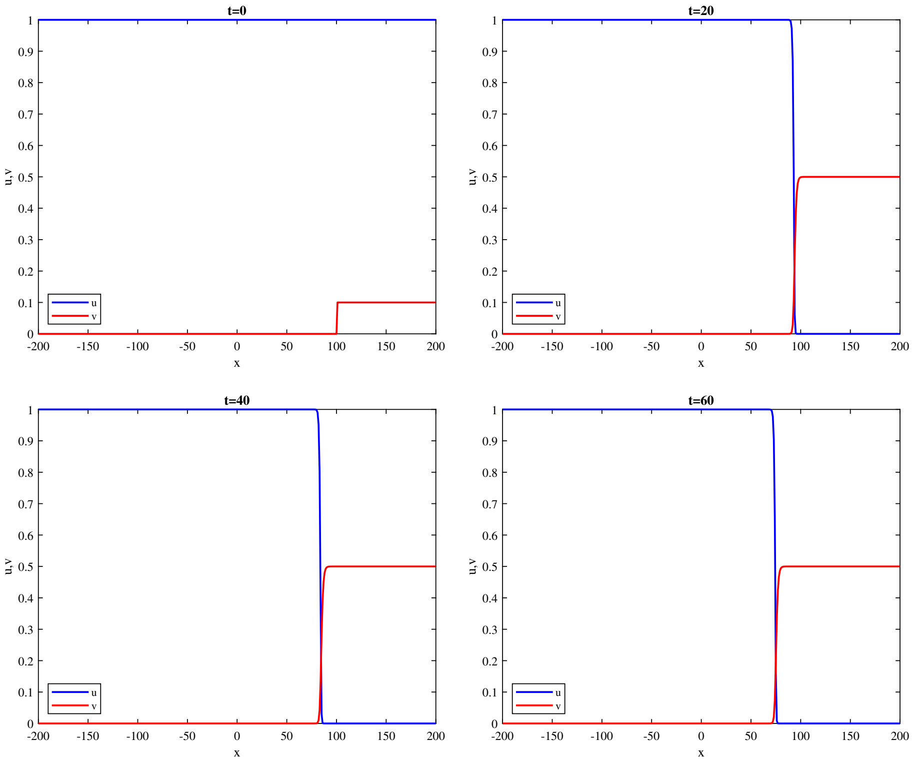

Fig. 3. Traveling wave of system (5.1) at different times using the parameters $d = 1$, $s = 0.5$, $e_1 = 1.2$, $e_2 = 0.5$ and $a = 15$.

With the aid of Lemma 5.3, we begin to prove the following theorem.

Theorem 5.4. *For $c \geq c^*$, there is a constant $\tilde{a} > 1/(1 + e_2)$ satisfying $S(\tilde{a}) = 0$ such that if $a < \bar{a} = \min\{\tilde{a}, 1/e_2\}$, then system (5.3) has a traveling wave connecting $(1, 0)$ and (u^*, v^*).*

Proof. Define

$$H(u,v) = \int_{u^*}^{u} \frac{(\eta + e_2)\eta - (u^* + e_2)u^*}{(\eta + e_2)\eta} d\eta + \varrho \int_{v^*}^{v} \frac{\eta - v^*}{\eta} d\eta,$$

where ϱ is a positive constant to be determined later. It is clear that

$$\begin{aligned} \partial_u H &= [(u + u^* + e_2)(u - u^*)]/[(u + e_2)u], \quad \partial_v H = \varrho(v - v^*)/v, \\ \partial_u^2 H &= [u^*(u^* + e_2)(2u + e_2)]/[(u + e_2)u]^2, \quad \partial_v^2 H = \varrho v^*/v^2. \end{aligned} \tag{5.4}$$

Then we define Lyapunov function $\mathcal{L} : \mathbb{R}^4 \to \mathbb{R}$ by

$$\mathcal{L}(u, w, v, y) = cH(u, v) - w\partial_u H - dy\partial_v H,$$

which is a continuous function with lower bound in Σ. A simple calculation yields

$$\begin{aligned}\mathcal{L}'(\chi(z)) &= (cw - w')\partial_u H + (cy - dy')\partial_v H - w^2\partial_u^2 H - y^2\partial_v^2 H \\ &= (u + u^* + e_2)(u - u^*)(1 - u - av)/(u + e_2) \\ &\quad + \varrho s(u + e_2 - v)(v - v^*)/(u + e_2) - w^2\partial_u^2 H - y^2\partial_v^2 H \\ &= \mathcal{H}(u, v)/(u + e_2) - w^2\partial_u^2 H - y^2\partial_v^2 H,\end{aligned}$$

where

$$\mathcal{H}(u, v) = (u + u^* + e_2)(u - u^*)(1 - u - av) + \varrho s(v - v^*)(u + e_2 - v).$$

To show that $\mathcal{L}'(\chi(z)) \le 0$ for $u, v > 0$, it is sufficient to prove $\mathcal{H}(u, v) \le 0$ owing to $\partial_u^2 H > 0$ and $\partial_v^2 H > 0$ for $u, v > 0$ from (5.4). Note that

$$\begin{aligned}1 - u - av &= -(u - u^*) - a(v - v^*), \\ u + e_2 - v &= (u - u^*) - (v - v^*),\end{aligned}$$

substituting these into $\mathcal{H}(u, v)$, we arrive at

$$\begin{aligned}\mathcal{H}(u, v) &= -(u + u^* + e_2)(u - u^*)^2 - \varrho s(v - v^*)^2 \\ &\quad + [\varrho s - a(u + u^* + e_2)](u - u^*)(v - v^*) \\ &= -\left[\sqrt{(u + u^* + e_2)}(u - u^*) - \frac{\varrho s - a(u + u^* + e_2)}{2\sqrt{(u + u^* + e_2)}}(v - v^*)\right]^2 \\ &\quad - \left(\varrho s - \frac{[\varrho s - a(u + u^* + e_2)]^2}{4(u + u^* + e_2)}\right)(v - v^*)^2.\end{aligned}$$

If $[\varrho s - a(u + u^* + e_2)]^2 - 4\varrho s(u + u^* + e_2) < 0$, that is

$$R(\varrho s) := (\varrho s)^2 - 2(a + 2)(u + u^* + e_2)\varrho s + a^2(u + u^* + e_2)^2 < 0, \tag{5.5}$$

then $\mathcal{H}(u, v) \le 0$ for $u, v > 0$. Obviously, $R(x)$ has two roots $0 < r_-(u) < r_+(u)$, where

$$\begin{aligned}r_-(u) &= (u + u^* + e_2)(\sqrt{a + 1} - 1)^2, \\ r_+(u) &= (u + u^* + e_2)(\sqrt{a + 1} + 1)^2.\end{aligned}$$

Equation (5.5) holds if and only if $r_-(u) < \varrho s < r_+(u)$ for $u \in (0, 1)$, which is equivalent to

$$\sup_{u\in(0,1)} r_-(u) < \inf_{u\in(0,1)} r_+(u)$$

by the arbitrariness of u. More precisely,

$$(1 + u^* + e_2)(\sqrt{a + 1} - 1)^2 < (u^* + e_2)(\sqrt{a + 1} + 1)^2.$$

Simplifying the above inequality, we have

$$(\sqrt{a + 1} - 1)^2 < 4(u^* + e_2)\sqrt{a + 1}.$$

Using the expression $u^* + e_2 = (1 + e_2)/(1 + a)$, we further get

$$(a + 2)\sqrt{a + 1} < 2(a + 1) + 4(1 + e_2),$$

which is equivalent to

$$S(a) = a^3 + a^2 - 16a(1 + e_2) - 16(1 + e_2)(2 + e_2) < 0.$$

From Lemma 5.3, we complete the proof. □

The last theorem establishes the existence of traveling waves connecting $(1, 0)$ and $(0, e_2)$ of system (5.3).

Theorem 5.5. *Assume that* $c \geq c^*$, *if* $ae_2^2 > 1 + e_2$, *then system* (5.3) *has a traveling wave connecting* $(1, 0)$ *and* $(0, e_2)$ *and satisfying* $v(z) \in \mathcal{A} \cup \mathcal{B}$ *and* $u'(z) < 0$ *over* $\mathbb{R}$.

Proof. Clearly, Assumptions 1.1 holds and $g(u) = a$. From Theorem 1.3, we complete the proof. □

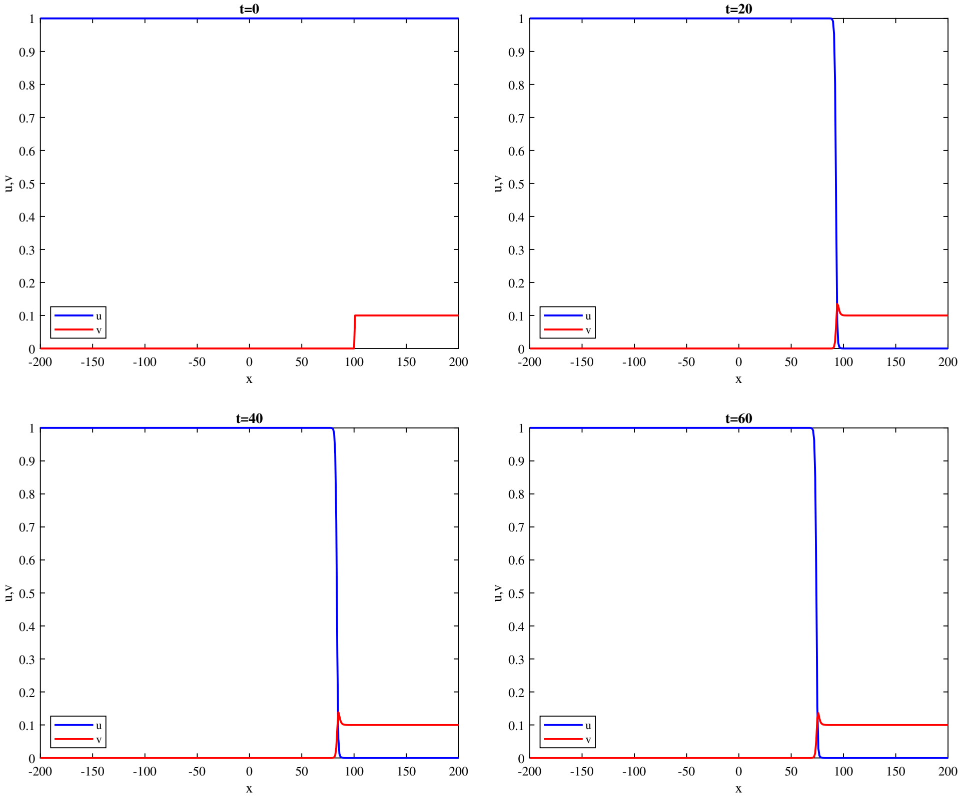

Fig. 4. Traveling wave of system (5.1) at different times using the parameters $d = 1$, $s = 0.5$, $a = 15$ and $e_2 = 0.1$.

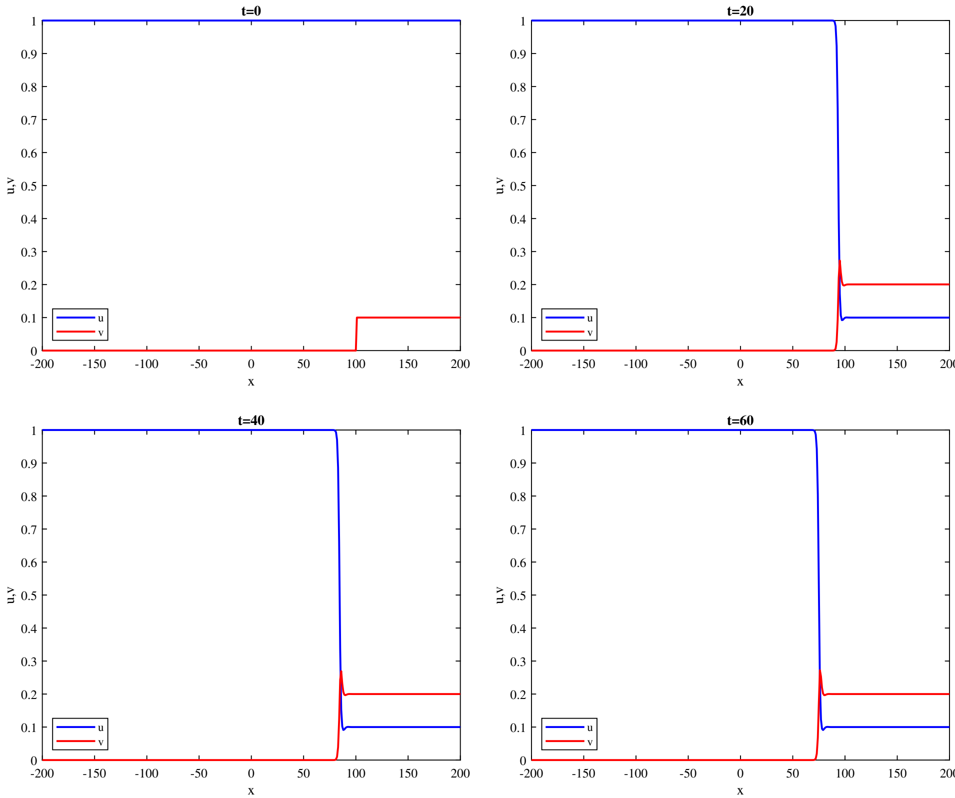

Fig. 5. Traveling wave of system (5.3) at different times using the parameters $d = 1$, $s = 0.5$, $a = 4.5$ and $e_2 = 0.1$.

We continue to illustrate our results through numerical simulations under the initial value (5.2). Let $d = 1$, $s = 0.5$, $e_2 = 0.1$ and $a = 15$, Fig. 4 depicts that u fades out. Moreover, let $d = 1$, $s = 0.5$ and $e_2 = 0.1$, then $\bar{a} \approx 4.5895$. We further choose $a = 4.5$, then $(u^*, v^*) = (0.1, 0.2)$. As shown in Fig. 5, system (5.3) has a traveling wave connecting $(1, 0)$ and (u^*, v^*).

6. Discussions

In this paper, by using the upper and lower solution method, we obtain the existence of traveling waves connecting $(1, 0)$ and E of a class of modified Leslie–Gower predator–prey systems with the aid of LaSalle's invariance principle.

To fulfill our purpose, we have made different technical assumptions

$$(P) : (u^* + \mu)f(u) - (u - u^*)(u + \mu)f'(u) > 0 \quad \text{for } u \in (0, 1)$$

in Theorem 1.2 and

$$(Q) : \min_{u\in[0,1]} g(u) > \frac{h(1)(h(1) + \mu)}{\mu^2 h(0)}$$

in Theorem 1.3, we now try to explain them in biological terms, respectively. First, the condition (Q) implies that an intense predator–prey relationship will eventually lead to the extinction of the prey, which follows the mechanism for selecting the superior and eliminating in the animal world, seeing Theorems 5.2 and 5.5. Next, let us focus on the condition (P), which seems to be interesting. We point out that the condition (P) clearly hold for $u \in (0, u^*]$ in Theorem 1.1, thus, one rewrite the condition (P) as

$$\frac{f'(u)}{f(u)} < \frac{(u^* + \mu)}{(u - u^*)(u + \mu)} \quad \text{for } u \in (u^*, 1).$$

Consider the simplest case in which the above inequality is true, this will be

$$\frac{f'(u)}{f(u)} < \kappa := \frac{(u^* + \mu)}{(1 - u^*)(1 + \mu)},$$

that is, $f(u) < f(u^*)e^{\kappa(u-u^*)}$ in $u \in (u^*, 1)$. Hence, when the favorite prey population size is low, *the generalist predators* focus on other prey species, and the predation relationship does not seem to affect the coexistence of two species in this case. When the favorite prey population size is high, predators will switch to hunting it, because it becomes profitable for them to do so, but the predator and the prey can still coexist in the long term as long as the predation effect is less than the exponential velocity. Finally, from results of Theorems 1.2 and 1.3, two open questions arise in a natural way. One is whether a suitable Lyapunov function can be constructed to ensure that the semi-travelling wave converges to (u^*, v^*) when $h(u) \neq 1$. Another point we should note is that system (1.2) with Holling-III type functional response function does not meet the condition (Q), hence how do the different functional response functions affect the propagation of the population in such a biological background?

Appendix

In this section, we explain how to calculate the upper solution and the lower solution in Sec. 3.

Case 1. $c > c^*$. First, since $\overline{u} \equiv 1$, $\underline{v} \geq 0$ and $p(1) = 0$, we have for $z \in \mathbb{R}$

$$\mathcal{U}(\overline{u}, \underline{v}) = -f(1)\underline{v} \leq 0.$$

Next, for $z > z_1$, it follows from $\underline{u} = 0$ that $\mathcal{U}(\underline{u}, \overline{v}) = 0$. For $z \leq z_1 (z_1 < 0)$, since $\underline{u} = 1 - \sigma e^{\beta z}$ and $\overline{v} = q(1)e^{\lambda_1 z}$, we have

$$\begin{aligned}\mathcal{U}(\underline{u}, \overline{v}) &= e^{\beta z}[\beta\sigma(c - \beta) - f(1 - \sigma e^{\beta z})q(1)e^{(\lambda_1 - \beta)z}] + f(1 - \sigma e^{\beta z})p(1 - \sigma e^{\beta z}) \\ &\geq e^{\beta z}[\beta\sigma(c - \beta) - f(1)q(1)] > 0\end{aligned}$$

by the choice of β and σ. For $\mathcal{V}(\overline{u}, \overline{v})$, we know that $\overline{v} = q(1)$ for $z > 0$, then

$$\mathcal{V}(\overline{u}, \overline{v}) = \mathcal{V}(1, q(1)) = sq(1)\left(1 - \frac{q(1)}{q(1)}\right) = 0.$$

For $z \le 0$, $\overline{v} = q(1)e^{\lambda_1 z}$. From the definition of λ_1, we have

$$\begin{aligned}\mathcal{V}(\overline{u},\overline{v}) &= q(1)d\lambda_1^2 e^{\lambda_1 z} - q(1)c\lambda_1 e^{\lambda_1 z} + sq(1)e^{\lambda_1 z} - sq(1)e^{2\lambda_1 z} \\ &= P(c,\lambda_1)q(1)e^{\lambda_1 z} - sq(1)e^{2\lambda_1 z} \\ &= -sq(1)e^{2\lambda_1 z} \le 0.\end{aligned}$$

Finally, for $z > z_2$, $\mathcal{V}(\underline{u},\underline{v}) = 0$ since $\underline{v} = 0$. Note that $\underline{v} = q(1)e^{\lambda_1 z}(1 - re^{\varepsilon z})$ and $\overline{v} = q(1)e^{\lambda_1 z}$ for $z \le z_2 (z_2 < 0)$, we have for $z \le z_2$

$$\begin{aligned}\underline{v} &= \overline{v} - rq(1)e^{(\lambda_1+\varepsilon)z}, \\ \underline{v}' &= \overline{v}' - rq(1)(\lambda_1+\varepsilon)e^{(\lambda_1+\varepsilon)z}, \\ \underline{v}'' &= \overline{v}'' - rq(1)(\lambda_1+\varepsilon)^2 e^{(\lambda_1+\varepsilon)z}.\end{aligned}$$

Then by the choice of ε and r, we further deduce that

$$\begin{aligned}\mathcal{V}(\underline{u},\underline{v}) &= d\overline{v}'' - drq(1)(\lambda_1+\varepsilon)^2 e^{(\lambda_1+\varepsilon)z} - c\overline{v}' + crq(1)(\lambda_1+\varepsilon)e^{(\lambda_1+\varepsilon)z} \\ &\quad + s\overline{v} - srq(1)e^{(\lambda_1+\varepsilon)z} - \frac{s(\overline{v} - rq(1)e^{(\lambda_1+\varepsilon)z})^2}{q(\underline{u})} \\ &\ge q(1)e^{\lambda_1 z}(d\lambda_1^2 - c\lambda_1 + s) - \frac{s\overline{v}^2}{q(0)} \\ &\quad -rq(1)e^{(\lambda_1+\varepsilon)z}[d(\lambda_1+\varepsilon)^2 - c(\lambda_1+\varepsilon) + s] \\ &= q(1)e^{\lambda_1 z}P(c,\lambda_1) - rq(1)e^{(\lambda_1+\varepsilon)z}P(c,\lambda_1+\varepsilon) - \frac{s\overline{v}^2}{q(0)} \\ &= -rq(1)e^{(\lambda_1+\varepsilon)z}P(c,\lambda_1+\varepsilon) - \frac{s\overline{v}^2}{q(0)} \\ &\ge q(1)e^{(\lambda_1+\varepsilon)z}\left[-rP(c,\lambda_1+\varepsilon) - \frac{sq(1)}{q(0)}e^{(\lambda_1-\varepsilon)z}\right] \\ &\ge q(1)e^{(\lambda_1+\varepsilon)z}\left[-rP(c,\lambda_1+\varepsilon) - \frac{sq(1)}{q(0)}\right] > 0.\end{aligned}$$

Case 2. $c = c^*$**.** First, as before, we have

$$\begin{aligned}&\mathcal{U}(\overline{u},\underline{v}) = -f(1)\underline{v} \le 0 \quad \text{for } z \in \mathbb{R}, \\ &\mathcal{U}(\underline{u},\overline{v}) = 0 \quad \text{for } z > z_1, \\ &\mathcal{V}(\overline{u},\overline{v}) = 0 \quad \text{for } z > -2/\lambda, \\ &\mathcal{V}(\underline{u},\underline{v}) = 0 \quad \text{for } z > z_2.\end{aligned}$$

Let us go on to proving the remaining inequalities. For $\mathcal{U}(\underline{u},\overline{v})$, when $z \le z_1(z_1 < -2/\lambda)$, we have $\underline{u} = 1 - \sigma e^{\beta z}$ and $\overline{v} = -hq(1)ze^{\lambda z}$. Note that for $z \in \mathbb{R}$

$$ze^{(\lambda-\beta)z} \ge -\frac{1}{(\lambda-\beta)e},$$

after a simple computation, we arrive at

$$\begin{aligned}
\mathcal{U}(\underline{u},\overline{v}) &= e^{\beta z}[\beta\sigma(c-\beta) + f(1-\sigma e^{\beta z})hq(1)ze^{(\lambda-\beta)z}] + f(1-\sigma e^{\beta z})p(1-\sigma e^{\beta z})\\
&\ge e^{\beta z}[\beta\sigma(c-\beta) + f(1-\sigma e^{\beta z})hq(1)ze^{(\lambda-\beta)z}]\\
&\ge e^{\beta z}[\beta\sigma(c-\beta) + f(1)hq(1)ze^{(\lambda-\beta)z}]\\
&\ge e^{\beta z}\left[\beta\sigma(c-\beta) - \frac{f(1)hq(1)}{(\lambda-\beta)e}\right] > 0
\end{aligned}$$

by the choice of β and σ. Next, for $\mathcal{V}(\overline{u},\overline{v})$, it is easy to see that if $z < -2/\lambda$, then

$$\overline{v}' = -hq(1)(1+\lambda z)e^{\lambda z}, \quad \overline{v}'' = -hq(1)\lambda(2+\lambda z)e^{\lambda z}.$$

Then we deduce that for $z < -2/\lambda$

$$\begin{aligned}
\mathcal{V}(\overline{u},\overline{v}) &\le d\overline{v}'' - c\overline{v}' + s\overline{v}\\
&= -dhq(1)\lambda(2+\lambda z)e^{\lambda z} + chq(1)(1+\lambda z)e^{\lambda z} - shq(1)ze^{\lambda z}\\
&= -hq(1)e^{\lambda z}[P(c,\lambda)z + (2d\lambda - c)] = 0.
\end{aligned} \tag{A.1}$$

Finally, for $z \le z_2(z_2 < -2/\lambda)$, $\overline{v} = -hq(1)ze^{\lambda z}$, we have

$$\underline{v} = q(1)e^{\lambda z}(-hz - r(-z)^{1/2}) = \overline{v} - rq(1)(-z)^{1/2}e^{\lambda z}.$$

Direct calculation gives

$$\underline{v}' = \overline{v}' + rq(1)e^{\lambda z}\left[\frac{1}{2}(-z)^{-1/2} - \lambda(-z)^{1/2}\right],$$

$$\underline{v}'' = \overline{v}'' + rq(1)e^{\lambda z}\left[\lambda(-z)^{-1/2} + \frac{1}{4}(-z)^{-3/2} - \lambda^2(-z)^{1/2}\right].$$

Substituting the above equalities into $\mathcal{V}(\underline{u},\underline{v})$, we obtain

$$\begin{aligned}
\mathcal{V}(\underline{u},\underline{v}) = d\overline{v}'' &+ drq(1)e^{\lambda z}\left[\lambda(-z)^{-1/2} + \frac{1}{4}(-z)^{-3/2} - \lambda^2(-z)^{1/2}\right] - c\overline{v}'\\
&-crq(1)e^{\lambda z}\left[\frac{1}{2}(-z)^{-1/2} - \lambda(-z)^{1/2}\right] + s\overline{v} - srq(1)(-z)^{1/2}e^{\lambda z} - \frac{s\underline{v}^2}{q(\underline{u})}.
\end{aligned}$$

From (A.1), we have for $z < -2/\lambda$

$$d\overline{v}'' - c\overline{v}' + s\overline{v} = 0.$$

Since $q(u)$ is an increasing function for $u > 0$, by the choice of r, we deduce that

$$\begin{aligned}
\mathcal{V}(\underline{u}, \underline{v}) &= drq(1)e^{\lambda z}\left[\lambda(-z)^{-1/2} + \frac{1}{4}(-z)^{-3/2} - \lambda^2(-z)^{1/2}\right] \\
&\quad - crq(1)e^{\lambda z}\left[\frac{1}{2}(-z)^{-1/2} - \lambda(-z)^{1/2}\right] - srq(1)(-z)^{1/2}e^{\lambda z} - \frac{s\underline{v}^2}{q(\underline{u})}, \\
&= q(1)e^{\lambda z}\left[\frac{1}{4}dr(-z)^{-3/2} + r(-z)^{-1/2}\left(d\lambda - \frac{c}{2}\right) - r(-z)^{1/2}P(c,\lambda)\right] - \frac{s\underline{v}^2}{q(\underline{u})} \\
&\geq q(1)e^{\lambda z}\left[\frac{1}{4}dr(-z)^{-3/2} - \frac{sh^2q(1)}{q(0)}z^2e^{\lambda z}\right] \\
&= q(1)e^{\lambda z}(-z)^{-3/2}\left[\frac{1}{4}dr - \frac{sh^2q(1)}{q(0)}(-z)^{7/2}e^{\lambda z}\right] \\
&\geq q(1)e^{\lambda z}(-z)^{-3/2}\left[\frac{1}{4}dr - \frac{sh^2q(1)}{q(0)}\left(\frac{7}{2e\lambda}\right)^{7/2}\right] > 0
\end{aligned}$$

by taking advantage of the fact that for $z \leq 0$

$$(-z)^{7/2}e^{\lambda z} \leq \left(\frac{7}{2e\lambda}\right)^{7/2}.$$

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