The Spreading Speed and the Existence of Planar Waves for a Class of Predator-prey System with Nonlocal Diffusion

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Abstract. In this paper, we study a predator-prey system with general response function and nonlocal diffusion in high dimensional space and investigate the propagation properties of its solution. More precisely, we study the invasion speed of the predator into habitat of the aborigine prey and obtain the existence of the planar waves by constructing the upper and lower solutions. Finally, we present some numerical simulations to support our results.

1. Introduction

In recent years, some nonlocal diffusion population dynamics models have been established in order to interpret spatial variation of interactions between different populations, see [5,11]. The study of dynamics of predator-prey interactions plays an important role in the theoretical ecology [8]. Ducrot et al. [6] discussed the asymptotic speed of propagation and minimum wave speed of a predator-prey model with Holling type I response function in one-dimensional space. In this paper, we consider a class of predator-prey model with general response functions, which is more corresponding with practical biological significance. The predator-prey model with general functional response and nonlocal dispersal in high dimensional space can be expressed as follows:

$$
(1.1) \quad
\begin{cases}
\frac{\partial U}{\partial t}(x,t) = d_1 \mathcal{N}_1[U(\,\cdot\,,t)](x) + r_1 U(x,t)(1 - U(x,t)) \\
\qquad\quad - f(U(x,t))V(x,t), \\
\frac{\partial V}{\partial t}(x,t) = d_2 \mathcal{N}_2[V(\,\cdot\,,t)](x) + \beta f(U(x,t))V(x,t) \\
\qquad\quad - r_2 V(x,t)(1 + V(x,t)),
\end{cases}
\quad x \in \mathbb{R}^N,\ t > 0.
$$

The system (1.1) is supplemented by the initial conditions:

$$
(1.2) \qquad U(x,0) = 1, \quad V(x,0) = v_0(x), \quad x \in \mathbb{R}^N,
$$

where v_0 is a nonnegative continuous function with nonempty compact support. In system (1.1), $U = U(x,t)$ and $V = V(x,t)$ are the population densities of prey and predator species at time $t > 0$ and the spatial position $x \in \mathbb{R}^N$, respectively. Here the dynamics of the prey population follow a logistic growth with a normalized (to one) carrying capacity and r_1 denotes its intrinsic growth rate. $f(U)$ is the functional response to the predator population. The constant β is a conversion rate, r_2 is the death rate of predator species, $r_1 > 0$, $r_2 > 0$, and $\beta > 0$. The term $d_i \mathcal{N}_i$ $(i = 1, 2)$ describes the spatial dispersal of the individual. Here, $d_1 > 0$ and $d_2 > 0$ are the diffusion coefficients for prey and predator species, respectively. And $\mathcal{N}_i$ $(i = 1, 2)$ is the linear nonlocal diffusion operators defined by

$$\mathcal{N}_i[\varphi](x) := (J_i * \varphi)(x) - \varphi(x) = \int_{\mathbb{R}^N} J_i(x - y)\varphi(y)\, \mathrm{d}y - \varphi(x),$$

where the symbol $*$ denotes the convolution product with respect to the spatial variable, and J_1, J_2 are probability kernel functions satisfying the following assumptions:

Assumption 1.1. The kernel function $J_i \colon \mathbb{R}^N \to \mathbb{R}$ $(i = 1, 2)$ is a nonnegative radial continuous function and satisfies $\int_{\mathbb{R}^N} J_i(x)\, \mathrm{d}x = 1$, $J_i(x) = J_i(-x)$ for all $x \in \mathbb{R}^N$ and there exists $\lambda_0 \in (0, +\infty]$ such that $\int_{\mathbb{R}^N} J_i(x)\mathrm{e}^{\lambda x_1}\mathrm{d}x < \infty$ for any $\lambda \in [0, \lambda_0)$.

To address more specifically the spreading speed and planar wave solutions of the system (1.1) with initial value conditions (1.2), we shall assume throughout this paper that the general functional response function satisfies certain assumptions, as follows:

Assumption 1.2. The function $f \colon [0, \infty) \to [0, \infty)$ is of class C^1 and satisfies:

(H1) $f(0) = 0$ and $f(x) > 0$ for any $x > 0$;

(H2) $f'(x) > 0$ for any $x \in [0, +\infty)$;

(H3) $\frac{f(x)}{x}$ is continuous and monotonely nonincreasing for $x > 0$.

Throughout this work, we shall assume the following parameter conditions:

(1.3) $\beta f(1) > r_2, \quad a(\beta f(1) - r_2) < r_1 r_2,$

where a is a positive constant satisfying $a \geq f'(0)$.

It is well known that local diffusion models are often used to describe the spread of infectious diseases in space and the invasion of populations in biology. However, this local diffusion model can only be used to describe the diffusion phenomenon near a small area. In the actual environment, the movements of individuals are usually random and free [3, 14]. Therefore, the nonlocal diffusion model is more realistic with the biological

environment. In the theory of reaction-diffusion equations, asymptotic speed of propagation and planar wave solutions play an important role in describing biological invasion and disease transmission.

Many scholars have conducted a lot of research on spreading speed and traveling wave solutions in one-dimensional space, see $[3, 7, 12, 14, 17, 18]$. In this paper, inspired by the processing of kernel functions in $[1, 4, 15]$, we mainly consider the spreading speed and planar wave solutions of predator-prey model with nonlocal diffusion and general response function in high-dimensional space. First, we consider the spreading speed of the predatory into habitat of the aborigine prey for model (1.1) with initial datum (1.2). Then, we consider the existence of the planar wave solutions which connects the predator-free state $(1, 0)$ with nontrivial state of the system (1.1). We will obtain that, under certain conditions, there exists $c^* > 0$ such that for $c \geq c^*$, system (1.1) admits invasion planar wave solutions with wave speed c; while for $0 < c < c^*$, system (1.1) has no invasion planar waves with wave speed c. We show that the spreading speed of the predatory is the minimal wave speed c^* of the invasion planar waves.

This paper is organized as follows. In the next section, we establish some preliminary results. In Section 3, we give the proof of the spreading speed by using the comparison principle. In Section 4, we firstly present the proof of the existence of the planar wave solutions by constructing appropriate upper and lower solutions under the assumption of the compactly supported for the kernel function J_2. Then, we obtain the nonexistence of the invasion planar wave solutions. Finally, we have the result that the spreading speed of the predatory is the same as the minimal wave speed c^*. In Section 5, we present some numerical simulations to support our results.

2. Preliminaries

Let
$$X = \{w(x) \mid w(x) \colon \mathbb{R}^N \to \mathbb{R} \text{ is bounded and uniformly continuous}\}$$

with norm
$$\|w\|_X = \sup_{x \in \mathbb{R}^N} |w(x)|,$$

thus $(X, \|\cdot\|_X)$ is a Banach space. The positive cone X^+ is defined by

$$X^+ = \{w \in X : w(x) \geq 0, \, \forall\, x \in \mathbb{R}^N\}.$$

Furthermore, for any constant $d > 0$, let

$$X_d = \{w \in X : 0 \leq w(x) \leq d, \, \forall\, x \in \mathbb{R}^N\}.$$

Set the order of the space $X^2 = X \times X$ as follows:

$$\underline{w} \le \overline{w} \quad \Longleftrightarrow \quad \underline{w}_i(x) \le \overline{w}_i(x), \quad x \in \mathbb{R}^N, \ i = 1, 2$$

for any $\underline{w} = (\underline{w}_1(x), \underline{w}_2(x))$ and $\overline{w} = (\overline{w}_1(x), \overline{w}_2(x)) \in X^2$.

We define the set $X_H^2 \subset X^2$ by

$$X_H^2 = \{(w_1, w_2) \in X^2 : 0 \le w_1 \le 1 \text{ and } 0 \le w_2 \le \alpha\},$$

where $\alpha := \frac{\beta}{r_2} f(1) - 1$. Our initial datum will always be chosen in this set. Here, we point out that under Assumption 1.2, the set X_H^2 is positively invariant under (1.1), therefore this system generates a strongly continuous nonlinear semiflow, denoted by $\mathbb{S}$, i.e., $\mathbb{S} = \{S(t) \colon X_H^2 \to X_H^2\}_{t \ge 0}$. In particular, this means that problem (1.1) with initial conditions (1.2) admits a unique globally defined solution (U, V) with

$$(U, V) \in C^1([0, \infty), X^2) \quad \text{and} \quad (U, V)(\,\cdot\,, t) \in X_H^2, \quad \forall t \ge 0.$$

Since (U, V) is bounded from $[0, \infty)$ into X^2, it follows from (1.1) that the time derivative of (U, V) is also bounded from $[0, \infty)$ into X^2. Hence, for each given initial data $v_0 \in X_\alpha$, the corresponding solution $(U, V) = (U, V)(x, t)$ of system (1.1) with (1.2) is uniformly continuous on $\mathbb{R}^N \times [0, \infty)$. Furthermore, if v_0 admits a nonempty compact support, then $V(t, x) > 0$ for all $t > 0$ and $x \in \mathbb{R}^N$.

For our purpose, we would firstly like to present a comparison principle for the following equation with constant coefficients:

$$(2.1) \qquad \begin{cases} \frac{\partial w(x,t)}{\partial t} = d\mathcal{N}[w(\,\cdot\,, t)](x) + rw(x, t)(s - w(x, t)), & x \in \mathbb{R}^N, \ t > 0, \\ w(x, 0) = \chi(x), & x \in \mathbb{R}^N, \end{cases}$$

where $\mathcal{N}[w] := J * w - w$, kernel J satisfies Assumption 1.1 and the initial data $\chi \in X_s$ admits a nonempty compact support. The parameters satisfy $d > 0$, $r > 0$ and $s > 0$. When $N = 1$, Jin and Zhao [10] studied a scalar periodic logistic equation with nonlocal diffusion and established a comparison principle. To our knowledge, a comparison principle for general nonlocal diffusion equation in high dimensional space can not be found easily. For reader's convenience, we provide a proof for such equation.

Proposition 2.1. *Let w be a solution of (2.1) with $w(\,\cdot\,, t) \in X_s$ for all $t > 0$ for a given $\chi \in X_s$. If $z(\,\cdot\,, t) \in X_s$ and $z(x, t)$ satisfies*

$$\begin{cases} \frac{\partial z(x,t)}{\partial t} \ge d\mathcal{N}[z(\,\cdot\,, t)](x) + rz(x, t)(s - z(x, t)), & x \in \mathbb{R}^N, \ t > 0, \\ z(x, 0) \ge \chi(x), & x \in \mathbb{R}^N, \end{cases}$$

then $z(x, t) \ge w(x, t)$ for all $x \in \mathbb{R}^N$, $t > 0$. Similar result holds for the reverse inequality.

Proof. We prove the proposition by modifying some arguments of [10, Theorem 2.3]. Define $\kappa(x,t) = z(x,t) - w(x,t)$ for all $x \in \mathbb{R}^N$, $t > 0$, and $\underline{\kappa}(t) = \inf_{x \in \mathbb{R}^N} \kappa(x,t)$ for all $t \geq 0$. Clearly, $\underline{\kappa}(t)$ is continuous in t and $\underline{\kappa}(0) \geq 0$. We prove that $\underline{\kappa}(t) \geq 0$ for all $t \geq 0$. Indeed, by contradiction, we assume that for $\delta > 0$, there exists $t_0 > 0$ such that $\underline{\kappa}(t_0) < 0$ and

$$(2.2) \qquad \underline{\kappa}(t_0)e^{-\delta t_0} = \min_{t \in [0,t_0]} \underline{\kappa}(t)e^{-\delta t} < \underline{\kappa}(\tau)e^{-\delta\tau}, \quad \forall\, \tau \in [0, t_0).$$

It follows that there exists a sequence of points $\{x_k\}_{k=1}^\infty$ such that $\kappa(x_k, t_0) < 0$ for all $k \geq 1$ and $\lim_{k\to\infty} \kappa(x_k, t_0) = \underline{\kappa}(t_0)$. Let $\{t_k\}_{k=1}^\infty \subseteq [0, t_0]$ be a sequence such that

$$(2.3) \qquad \kappa(x_k, t_k)e^{-\delta t_k} = \min_{t \in [0,t_0]} \kappa(x_k, t)e^{-\delta t}.$$

Let $m_\varepsilon = \min_{t \in [0, t_0 - \varepsilon]} \underline{\kappa}(t)^{e^{-\delta t}}$ for any $\varepsilon \in (0, t_0)$. Using (2.2), we obtain

$$\lim_{k\to\infty} \kappa(x_k, t_0)e^{-\delta t_0} = \underline{\kappa}(t_0)e^{-\delta t_0} < m_\varepsilon.$$

Thus, there exists an integer K_ε such that for all $k \geq K_\varepsilon$,

$$\kappa(x_k, t_0)e^{-\delta t_0} < m_\varepsilon \leq \underline{\kappa}(t)e^{-\delta t} \leq \kappa(x_k, t)e^{-\delta t}, \quad \forall\, t \in [0, t_0 - \varepsilon].$$

By (2.3), we have

$$\kappa(x_k, t_k)e^{-\delta t_k} = \min_{t \in [0,t_0]} \kappa(x_k, t)^{e^{-\delta t}} \leq \kappa(x_k, t_0)e^{-\delta t_0},$$

thus, $t_k \in [t_0 - \varepsilon, t_0]$ for all $k \geq K_\varepsilon$. Then we deduce that $\lim_{k\to\infty} t_k = t_0$. Due to

$$\kappa(x_k, t_0)e^{-\delta t_0} \geq \kappa(x_k, t_k)e^{-\delta t_k} \geq \underline{\kappa}(t_k)e^{-\delta t_k} \geq \underline{\kappa}(t_0)e^{-\delta t_0},$$

we get

$$\kappa(x_k, t_0)e^{-\delta(t_0 - t_k)} \geq \kappa(x_k, t_k) \geq \underline{\kappa}(t_0)e^{-\delta(t_0 - t_k)}.$$

Letting $k \to \infty$, we have $\lim_{k\to\infty} \kappa(x_k, t_k) = \underline{\kappa}(t_0)$. Then (2.3) implies that

$$0 \geq \left.\frac{\partial(\kappa(x_k, t)e^{-\delta t})}{\partial t}\right|_{t = t_k^-} = e^{-\delta t_k}\left(\frac{\partial\kappa(x_k, t_k)}{\partial t} - \delta\kappa(x_k, t_k)\right).$$

Thus

$$\frac{\partial\kappa(x_k, t_k)}{\partial t} \leq \delta\kappa(x_k, t_k).$$

The function $g \colon \mathbb{R} \to \mathbb{R}$ is denoted

$$g(l) := rl(s - l),$$

where $l := l(x,t)$, $\forall\, x \in \mathbb{R}^N$, $t \geq 0$. By the definition of g, for every $M_0 > 0$, there exists constant $L > 0$ such that

$$(2.4) \qquad |g(z) - g(w)| \leq L|z - w|, \quad \forall\, z, w \in [0, M_0].$$

Because $\kappa(x_k, t_k) < 0$ and by (2.4), it follows that

$$
\begin{aligned}
\frac{\partial \kappa(x_k, t_k)}{\partial t} &= \frac{\partial z(x_k, t_k)}{\partial t} - \frac{\partial w(x_k, t_k)}{\partial t} \\
&\geq d\mathcal{N}[\kappa(\,\cdot\,, t_k)](x_k) + g(z(x_k, t_k)) - g(w(x_k, t_k)) \\
&\geq d\left[\int_{\mathbb{R}^N} J(x_k - y)\kappa(y, t_k)\,\mathrm{d}y - \kappa(x_k, t_k)\right] + L\kappa(x_k, t_k).
\end{aligned}
$$

Then

$$
\begin{aligned}
0 &\leq \frac{\partial \kappa(x_k, t_k)}{\partial t} - L\kappa(x_k, t_k) - d\left[\int_{\mathbb{R}^N} J(x_k - y)\kappa(y, t_k)\,\mathrm{d}y - \kappa(x_k, t_k)\right] \\
&\leq (\delta - L + d)\kappa(x_k, t_k) - d\underline{\kappa}(t_k).
\end{aligned}
$$

Letting $k \to \infty$, we obtain $0 \leq (\delta - L)\underline{\kappa}(t_0)$. For $\delta > L$, it follows that $\underline{\kappa}(t_0) \geq 0$, a contradiction. Therefore, we get $\underline{\kappa}(t) \geq 0$ for any $t \geq 0$. Thus, we obtain $\kappa(x, t) \geq 0$, i.e.,

$$
z(x, t) \geq w(x, t) \quad \text{for all } x \in \mathbb{R}^N,\ t \geq 0.
$$

We complete the proof. $\square$

Remark 2.2. Through the proof of Proposition 2.1, we can generalize its conclusion to a more general system:

$$
\begin{cases}
\frac{\partial w(x,t)}{\partial t} = d\mathcal{N}[w(\,\cdot\,, t)](x) + w(x, t)F(w(x, t)), & x \in \mathbb{R}^N,\ t > 0, \\
w(x, 0) = \chi(x), & x \in \mathbb{R}^N,
\end{cases}
$$

where $F \in C^1(\mathbb{R}, \mathbb{R})$, $F_w < 0$ for all $w \in \mathbb{R}$, we have $z(x, t) \geq w(x, t)$ for all $x \in \mathbb{R}^N$, $t > 0$, where $z(x, t)$ satisfies

$$
\begin{cases}
\frac{\partial z(x,t)}{\partial t} \geq d\mathcal{N}[z(\,\cdot\,, t)](x) + z(x, t)F(z(x, t)), & x \in \mathbb{R}^N,\ t > 0, \\
z(x, 0) \geq \chi(x), & x \in \mathbb{R}^N.
\end{cases}
$$

Moreover, Jin and Zhao [10] studied the spreading speed of a scalar periodic logistic equation with nonlocal diffusion in one-dimensional space. Diekmann [4] investigated the spreading speed of the nonlinear integral equation of mixed Volterra–Fredholm type in high-dimensional case. Ma [13] explored the asymptotic speed of spread of a general nonlocal diffusion equation in n-dimensional case. Based on their research, we conclude that system (2.1) has the following spreading speed.

Proposition 2.3. *Let w be a solution of* (2.1) *with $w(\,\cdot\,, t) \in X_s$ for all $t > 0$ for a given $\chi \in X_s$ and $\bar{c} := \inf_{0 < \lambda < \lambda_0} \frac{d\left[\int_{\mathbb{R}^N} J(x)\mathrm{e}^{\lambda x_1}\,\mathrm{d}x - 1\right] + rs}{\lambda} > 0$. Then the following statements are valid.*

(i) *For any $c > \bar{c}$, if χ has a nonempty compact support, then*

(2.5)
$$\lim_{t \to \infty} \sup_{\|x\| > ct} w(x, t) = 0;$$

(ii) *For any $0 < c < \bar{c}$, if $\chi(\cdot) \not\equiv 0$, then*

$$\liminf_{t \to \infty} \inf_{\|x\| < ct} w(x, t) = s.$$

Since the kernel function J_2 satisfies Assumption 1.1, we define

(2.6)
$$c^* := \inf_{0 < \lambda < \lambda_0} \frac{d_2 \left[\int_{\mathbb{R}^N} J_2(x) e^{\lambda x_1} \, \mathrm{d}x - 1 \right] + \beta f(1) - r_2}{\lambda}.$$

3. Spreading speed

In this section, we will discuss the spreading speed of the predator for system (1.1) with initial values (1.2) and establish the following theorem.

Theorem 3.1. *Under the assumption (1.3), the constant c^*, defined in (2.6), corresponds to the spreading speed of the predator for system (1.1) with initial datum (1.2) and $0 \le v_0 \le \alpha$. This means that the density of the predator $V = V(x, t)$ satisfies the following statements.*

(1) *For any $c > c^*$, if v_0 is a nonzero compactly supported continuous function, then*

$$\lim_{t \to \infty} \sup_{\|x\| > ct} V(x, t) = 0;$$

(2) *For any $0 < c < c^*$, if $v_0(\cdot) \not\equiv 0$, then*

$$\liminf_{t \to \infty} \inf_{\|x\| < ct} V(x, t) > 0.$$

Proof. First, we prove that

(3.1)
$$\lim_{t \to \infty} \sup_{\|x\| > ct} V(x, t) = 0 \quad \text{for any } c > c^*.$$

Since $U \le 1$, using (H2), we get that $f(U) \le f(1)$ and the function V satisfies

$$\frac{\partial V(x, t)}{\partial t} \le d_2 \mathcal{N}_2[V(\cdot, t)](x) + \beta f(1) V(x, t) - r_2 V(x, t)(1 + V(x, t)), \quad x \in \mathbb{R}^N, \ t > 0.$$

By Proposition 2.1, for all $(x, t) \in \mathbb{R}^N \times [0, \infty)$, we have $V(x, t) \le \overline{V}(x, t)$, where $\overline{V}$ satisfies

$$\begin{cases} \frac{\partial \overline{V}(x,t)}{\partial t} = d_2 \mathcal{N}_2[\overline{V}(\cdot, t)](x) + \beta f(1) \overline{V}(x, t) - r_2 \overline{V}(x, t)(1 + \overline{V}(x, t)), \quad x \in \mathbb{R}^N, \ t > 0, \\ \overline{V}(x, 0) = v_0(x). \end{cases}$$

Hence, according to (2.5) in Proposition 2.3, we obtain (3.1).

Now we prove the statement (2), namely

$$\liminf_{t\to\infty}\inf_{\|x\|<ct} V(x,t) > 0 \quad \text{for any given } c \in (0, c^*).$$

According to (H2) and (H3), we deduce that $f(x) \leq ax$, where $a \geq f'(0) > 0$. Since $V(x,t) \leq \alpha$, the function U satisfies

$$\frac{\partial U(x,t)}{\partial t} \geq d_1\mathcal{N}_1[U(\,\cdot\,,t)](x) + U(x,t)(r_1 - a\alpha - r_1U(x,t)), \quad x \in \mathbb{R}^N,\ t > 0.$$

Hence, according to the second condition in (1.3), i.e., $a(\beta f(1) - r_2) < r_1r_2$ and $U(x,0) = 1$, we derive from Proposition 2.1 that

$$(3.2) \qquad U(x,t) \geq U_{\min} := 1 - \frac{a}{r_1r_2}(\beta f(1) - r_2), \quad \forall\, x \in \mathbb{R}^N,\ t \geq 0.$$

Also using (3.2), $U \leq 1$ and (H2), the function U satisfies

$$\frac{\partial U(x,t)}{\partial t} \geq d_1\mathcal{N}_1[U(\,\cdot\,,t)](x) + r_1U_{\min}(1 - U(x,t)) - f(1)V(x,t), \quad x \in \mathbb{R}^N,\ t > 0.$$

Because $U(x,0) = 1$, the function $1 - U(x,t)$ satisfies

$$(3.3)\ \ 1 - U(x,t) \leq \overline{u}(x,t) := f(1)\int_0^t \exp(-r_1U_{\min}(t-s))\exp((t-s)d_1\mathcal{N}_1)[V(\,\cdot\,,s)](x)\,\mathrm{d}s$$

for all $x \in \mathbb{R}^N$, $t \geq 0$. By (H3), we deduce that $f(U) \geq -f(1)\overline{u} + f(1)$. Plugging (3.3) into the V-equation in (1.1) yields

$$(3.4) \qquad \begin{aligned} \frac{\partial V(x,t)}{\partial t} &\geq d_2\mathcal{N}_2[V(\,\cdot\,,t)](x) \\ &\quad + V(x,t)[-\beta f(1)\overline{u}(x,t) - r_2V(x,t) - r_2 + \beta f(1)], \quad x \in \mathbb{R}^N,\ t > 0. \end{aligned}$$

Let $\epsilon \in (0, c^*)$ be given, $\delta_0 > 0$ small enough and $\delta_0 < \beta f(1) - r_2$ such that

$$\inf_{0<\lambda<\lambda_0} \frac{d_2\left[\int_{\mathbb{R}^N} J_2(x)e^{\lambda x_1}\,\mathrm{d}x - 1\right] + \beta f(1) - r_2 - \delta_0}{\lambda} > c^* - \epsilon.$$

Next, we claim there exist $M > 0$ and $\tau > 0$ large enough such that

$$(3.5) \qquad \beta f(1)\overline{u}(x,t) \leq \delta_0 + MV(x,t), \quad x \in \mathbb{R}^N,\ t \geq \tau.$$

Indeed, to prove the claim, it suffices to show that there exists some constant $M > 0$ such that, for any $t \geq \tau$ and $x \in \mathbb{R}^N$, we have

$$(3.6) \qquad \beta f(1)\overline{u}(x,t) \geq \delta_0 \quad \Longrightarrow \quad \beta f(1)\overline{u}(x,t) \leq \delta_0 + MV(x,t).$$

To the aim, we consider the strongly positive semigroup $\{T(t)\}_{t \geq 0}$, where

$$T(t) := \exp(td_1 \mathcal{N}_1), \quad t \geq 0.$$

According to [9], we know that the solution W of the problem

$$\frac{\partial w(x,t)}{\partial t} = d_1\{[J_1 * w(\,\cdot\,,t)](x) - w(x,t)\}, \quad t > 0 \text{ with } w(\,\cdot\,,0) = \delta_1,$$

where δ_1 denotes the Dirac mass at $x = 0$, can be decomposed as

$$W(x,t) = \mathrm{e}^{-d_1 t}\delta_1(x) + K(x,t), \quad x \in \mathbb{R}^N, \ t \geq 0,$$

where K is a nonnegative smooth function satisfying the estimate

$$(3.7) \qquad \int_{\mathbb{R}^N} K(x,t)\,\mathrm{d}x \leq 2, \quad \forall t \geq 0.$$

Thus, the semigroup $\{T(t)\}_{t \geq 0}$ can be expressed as

$$T(t)[\varphi](x) = \mathrm{e}^{-d_1 t}\varphi(x) + \int_{\mathbb{R}^N} K(x - y,t)\varphi(y)\,\mathrm{d}y, \quad \forall t \geq 0, \ \varphi \in X.$$

Using the above formula for $T(t)$, the function $\beta f(1)\overline{u}(x,t)$ can be decomposed as $\beta f(1)$ $\overline{u}(x,t) = W_1(x,t) + W_2(x,t)$, where

$$W_1(x,t) := a\beta f(1) \int_0^t \mathrm{e}^{-\beta_1(t-s)}V(x,s)\,\mathrm{d}s, \quad \beta_1 := d_1 + r_1 U_{\min} > 0,$$

$$W_2(x,t) := a\beta f(1) \int_0^t \int_{\mathbb{R}^N} \mathrm{e}^{-\beta_2(t-s)}K(x - y, t - s)V(y,s)\,\mathrm{d}y\mathrm{d}s, \quad \beta_2 := r_1 U_{\min} > 0.$$

Since $\delta_0 > 0$ is fixed, there exists $\tau > 0$ large enough such that

$$(3.8) \quad a\beta f(1)\alpha \int_0^{t-\tau} \mathrm{e}^{-\beta_1(t-s)}\,\mathrm{d}s \leq \frac{\delta_0}{4}, \quad 2a\beta f(1)\alpha \int_0^{t-\tau} \mathrm{e}^{-\beta_2(t-s)}\,\mathrm{d}s \leq \frac{\delta_0}{4}, \quad \forall t \geq \tau.$$

In order to prove (3.6), we consider $t_0 \geq \tau$ and $x_0 \in \mathbb{R}^N$ with $\beta f(1)\overline{u}(x_0,t_0) \geq \delta_0$. First, since $V \leq \alpha$, from (3.7), it follows that

$$\delta_0 \leq \beta f(1)\overline{u}(x_0,t_0) \leq a\beta f(1)\alpha \int_0^{t_0-\tau} \mathrm{e}^{-\beta_1(t_0-s)}\,\mathrm{d}s + 2a\beta f(1)\alpha \int_0^{t_0-\tau} \mathrm{e}^{-\beta_2(t_0-s)}\,\mathrm{d}s$$

$$(3.9) \qquad\qquad\qquad + a\beta f(1)\int_{t_0-\tau}^{t_0} \mathrm{e}^{-\beta_1(t_0-s)}V(x_0,s)\,\mathrm{d}s$$

$$+ a\beta f(1)\int_{t_0-\tau}^{t_0} \int_{\mathbb{R}^N} \mathrm{e}^{-\beta_2(t_0-s)}K(x_0 - y, t_0 - s)V(y,s)\,\mathrm{d}y\mathrm{d}s.$$

This combined with (3.8), yields

$$a\beta f(1)\int_0^\tau \mathrm{e}^{-\beta_1 l}V(x_0,t_0-l)\,\mathrm{d}l + a\beta f(1)\int_0^\tau \int_{\mathbb{R}^N} \mathrm{e}^{-\beta_2 l}K(y,l)V(x_0-y,t_0-l)\,\mathrm{d}y\mathrm{d}l \geq \frac{\delta_0}{2}.$$

Next, choose $r > 0$ large enough such that

$$a\beta f(1)\alpha \int_0^\tau \int_{\|y\| \geq r} \mathrm{e}^{-\beta_2 l} K(y,l)\,\mathrm{d}y\mathrm{d}l \leq \frac{\delta_0}{4}.$$

According to $V \leq \alpha$, it follows that

$$a\beta f(1) \int_0^\tau \mathrm{e}^{-\beta_1 l} V(x_0, t_0 - l)\,\mathrm{d}l + a\beta f(1) \int_0^\tau \int_{\|y\| \leq r} \mathrm{e}^{-\beta_2 l} K(y,l) V(x_0 - y, t_0 - l)\,\mathrm{d}y\mathrm{d}l \geq \frac{\delta_0}{4}.$$

Let $\eta > 0$ small enough such that

$$a\beta f(1)\alpha \left[\int_0^\eta \mathrm{e}^{-\beta_1 l}\,\mathrm{d}l + \int_0^\eta \int_{\|y\| \leq r} \mathrm{e}^{-\beta_2 l} K(y,l)\,\mathrm{d}y\mathrm{d}l \right] \leq \frac{\delta_0}{8}.$$

Thus, using the same argument as above, it follows that

$$a\beta f(1) \int_\eta^\tau \mathrm{e}^{-\beta_1 l} V(x_0, t_0 - l)\,\mathrm{d}l + a\beta f(1) \int_\eta^\tau \int_{\|y\| \leq r} \mathrm{e}^{-\beta_2 l} K(y,l) V(x_0 - y, t_0 - l)\,\mathrm{d}y\mathrm{d}l \geq \frac{\delta_0}{8}.$$

Setting $\theta > 0$ by

$$\frac{\delta_0}{8} = \theta \left[a\beta f(1) \int_\eta^\tau \mathrm{e}^{-\beta_1 l}\,\mathrm{d}l + a\beta f(1) \int_\eta^\tau \int_{\|y\| \leq r} \mathrm{e}^{-\beta_2 l} K(y,l)\,\mathrm{d}y\mathrm{d}l \right],$$

it follows that there are $l_0 \in [t_0 - \tau, t_0 - \eta]$ and $y_0 \in B_r(x_0)$ such that $V(y_0, l_0) \geq \theta$. Furthermore, since the function V is uniformly continuous on $\mathbb{R}^N \times [0, \infty)$, there exists $\rho > 0$ independent of (y_0, l_0) such that

$$V(y, l_0) \geq \frac{\theta}{2}, \quad \forall y \in \overline{B}_\rho(y_0),$$

where $\overline{B}$ represents the closure of B. Finally, we define a uniformly continuous function $Z_0 \leq \frac{\theta}{2}$ in $\mathbb{R}$ such that

$$Z_0(x) = \frac{\theta}{2}, \ \ \forall x \in \overline{B}_{\frac{\rho}{2}}(0), \quad Z_0(x) = 0, \ \ \forall x \in \mathbb{R}^N \setminus \overline{B}_\rho(0).$$

Clearly, the function V satisfies

$$\frac{\partial V(x,t)}{\partial t} \geq d_2 \mathcal{N}_2[V(\cdot, t)](x) - r_2(1 + \alpha)V(x,t), \quad x \in \mathbb{R}^N,\ t > 0.$$

Since $Z_0(x) \leq V(y_0 + x, l_0)$, according to the comparison principle, we obtain

$$Z(x,t) \leq V(y_0 + x, l_0 + t), \quad \forall x \in \mathbb{R}^N,\ t \geq 0,$$

where $Z(\cdot, t) = \mathrm{e}^{r_2(1+\alpha)t} \exp(td_2\mathcal{N}_2)[Z_0](\cdot)$. Clearly $Z(x,t) > 0$ for all $x \in \mathbb{R}^N,\ t > 0$. Thus, we get that

$$V(x_0, t_0) \geq Z(x_0 - y_0, t_0 - l_0) \geq \gamma := \min_{x \in \overline{B}_r(0),\ t \in [\eta, \tau]} Z(x,t) > 0.$$

Then, using (3.7), (3.8), (3.9) and $V \leq \alpha$, we have that

$$\beta f(1)\overline{u}(x_0, t_0) < \frac{\delta_0}{2} + 3a\beta f(1)\alpha\tau \leq \delta_0 + MV(x_0, t_0),$$

where we set $M = 3\gamma^{-1}a\beta f(1)\alpha\tau > 0$. Clearly, M is independent of (x_0, t_0). This completes the proof of the claim.

Inserting (3.5) into (3.4) yields

$$\frac{\partial V(x,t)}{\partial t} \geq d_2 \mathcal{N}_2[V(\,\cdot\,,t)](x) + V(x,t)[\beta f(1) - r_2 - \delta_0 - (r_2 + M)V(x,t)], \quad x \in \mathbb{R}^N, \ t > \tau.$$

According to the comparison principle, it follows that $V(x, t+\tau) \geq \underline{V}(x,t)$ for any $x \in \mathbb{R}^N$, $t \geq 0$, where the function $\underline{V}$ is a solution of the scalar logistic equation

$$\begin{cases} \frac{\partial \underline{V}(x,t)}{\partial t} = d_2 \mathcal{N}_2[\underline{V}(\,\cdot\,,t)](x) \\ \qquad\qquad + \underline{V}(x,t)[\beta f(1) - r_2 - \delta_0 - (r_2 + M)\underline{V}(x,t)], \quad x \in \mathbb{R}^N, \ t > \tau, \\ \underline{V}(x,0) \leq V(x,\tau). \end{cases}$$

Then, because $\epsilon > 0$ is small enough and Proposition 2.3(ii), we obtain that

$$\liminf_{t\to\infty} \inf_{\|x\|<(c^*-\epsilon)t} V(x,t) \geq \liminf_{t\to\infty} \inf_{\|x\|<(c^*-\epsilon)t} \underline{V}(x,t) = \frac{\beta f(1) - r_2 - \delta_0}{r_2 + M} > 0.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

4. Planar wave solutions and minimal wave speed

In this section, we mainly investigate the planar wave solutions for system (1.1) connecting the predator-free state to a nontrivial state. In Section 4.1, we give a general result. We prove the existence of planar wave solutions for system (1.1) if $c \geq c^*$ in Section 4.2. In Section 4.3, we show the nonexistence of planar wave solutions for system (1.1) if $c \in (0, c^*)$. In Section 4.4, we explain that the spreading speed of the predatory in Theorem 3.1 is the minimal wave speed c^*.

4.1. A general result

In this subsection, we give a general result about the planar wave solutions of system (1.1) with (1.2). A solution (U, V) to (1.1) is called a planar wave solutions of (1.1), if there exists a function pair (Φ, Ψ), the wave profile, such that $(U, V)(x, t) = (\Phi, \Psi)(\xi)$, $\xi := \nu \cdot x + ct$, where c is a nonnegative constant, $\nu \in \mathbb{R}^N$ is a fixed unit vector and $x \cdot \nu = \sum_{j=1}^{N} x_j \nu_j$. Here, we focus on the planar waves connecting the predator-free state $(1, 0)$ at $\xi = -\infty$ to a nontrivial state at $\xi = \infty$ in the sense that

$$\text{(4.1)} \qquad\qquad\qquad\qquad \liminf_{\xi\to\infty} \Phi(\xi) > 0, \quad \liminf_{\xi\to\infty} \Psi(\xi) > 0.$$

In fact, the "nontrivial state" at $\xi = \infty$ defined in the sense of (4.1) only means that the right tail of the planar wave is far from zero. It is difficult to describe the exact nontrivial state of the system due to the lack of comparison principle and the nonlocal diffusion of the system.

We make the following transformation:

$$u(x,t) = 1 - U(x,t), \quad v(x,t) = V(x,t), \quad \bar{a} = \frac{a}{r_1}.$$

With these notations, system (1.1) is rewritten as

(4.2)
$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = d_1 \mathcal{N}_1[u(\,\cdot\,,t)](x) - r_1 u(x,t)[1 - u(x,t)] + f(1 - u(x,t))v(x,t), \\ \frac{\partial v}{\partial t}(x,t) = d_2 \mathcal{N}_2[v(\,\cdot\,,t)](x) + r_2 v(x,t)[\bar{\beta} f(1 - u(x,t)) - 1 - v(x,t)]. \end{cases}$$

At this time condition (1.3) becomes

$$\bar{\beta} f(1) > 1, \quad \bar{a}(\bar{\beta} f(1) - 1) < 1.$$

To find a planar wave solutions of (1.1) is equivalent to finding a solution for system (4.2) in the form

$$u(x,t) = \phi(\xi), \quad v(x,t) = \psi(\xi), \quad \text{where } \xi := \nu \cdot x + ct$$

such that $(\phi, \psi)(-\infty) = (0,0)$ and

(4.3)
$$\limsup_{\xi \to \infty} \phi(\xi) < 1, \quad \liminf_{\xi \to \infty} \psi(\xi) > 0.$$

Since ν is a unit vector, we can rotate the coordinates so that $\nu = (1, 0, \ldots, 0)$ in the new coordinates. Due to the kernel function is a radial function, thus $(\phi(\xi), \psi(\xi))$ satisfies the following ordinary differential equation independently of the unit vector ν:

(4.4)
$$\begin{cases} c\phi'(\xi) = d_1 \widetilde{\mathcal{N}}_1[\phi](\xi) - r_1 \phi(\xi)[1 - \phi(\xi)] + f(1 - \phi(\xi))\psi(\xi), \\ c\psi'(\xi) = d_2 \widetilde{\mathcal{N}}_2[\psi](\xi) + r_2 \psi(\xi)[\bar{\beta} f(1 - \phi(\xi)) - 1 - \psi(\xi)], \end{cases}$$

where the linear operators $\widetilde{\mathcal{N}}_i$ are defined by

$$\widetilde{\mathcal{N}}_i[\varphi](\xi) := \int_{\mathbb{R}} \widetilde{J}_i(x_1) \varphi(\xi - x_1) \, \mathrm{d}x_1 - \varphi(\xi), \quad i = 1, 2,$$

and

$$\widetilde{J}_i(x_1) := \int_{\mathbb{R}^{N-1}} J_i(x_1, x_2, \ldots, x_N) \, \mathrm{d}x_2 \cdots \mathrm{d}x_N, \quad i = 1, 2.$$

Thus, according to the definition of $\mathcal{N}_i$, $\widetilde{\mathcal{N}}_i$ ($i = 1, 2$) and (2.6), we have

$$c^* = \inf_{0 < \lambda < \lambda_0} \frac{d_2 \left[\int_{\mathbb{R}} \widetilde{J}_2(x_1) \mathrm{e}^{\lambda x_1} \, \mathrm{d}x_1 - 1 \right] + \beta f(1) - r_2}{\lambda}.$$

We begin with the following definition.

Definition 4.1. A pair of functions $(\overline{\phi}, \overline{\psi}), (\underline{\phi}, \underline{\psi}) \in X_h^2$ is called a pair of super and sub solutions of (4.4) if $(\underline{\phi}, \underline{\psi}) \leq (\overline{\phi}, \overline{\psi})$ for all $\xi \in \mathbb{R}$ and the following inequalities

$$(4.5) \qquad c\overline{\phi}'(\xi) \geq d_1\widetilde{\mathcal{N}}_1[\overline{\phi}](\xi) - r_1\overline{\phi}(\xi)[1 - \overline{\phi}(\xi)] + f(1 - \overline{\phi}(\xi))\overline{\psi}(\xi),$$

$$(4.6) \qquad c\overline{\psi}'(\xi) \geq d_2\widetilde{\mathcal{N}}_2[\overline{\psi}](\xi) + r_2\overline{\psi}(\xi)[\overline{\beta}f(1 - \underline{\phi}(\xi)) - 1 - \overline{\psi}(\xi)],$$

$$(4.7) \qquad c\underline{\phi}'(\xi) \leq d_1\widetilde{\mathcal{N}}_1[\underline{\phi}](\xi) - r_1\underline{\phi}(\xi)[1 - \underline{\phi}(\xi)] + f(1 - \underline{\phi}(\xi))\underline{\psi}(\xi),$$

$$(4.8) \qquad c\underline{\psi}'(\xi) \leq d_2\widetilde{\mathcal{N}}_2[\underline{\psi}](\xi) + r_2\underline{\psi}(\xi)[\overline{\beta}f(1 - \overline{\phi}(\xi)) - 1 - \underline{\psi}(\xi)]$$

hold for all $\xi \in \mathbb{R} \setminus E$, where E denotes some finite set $E \subset \mathbb{R}$ and $X_h^2 := X_1 \times X_{\overline{\alpha}}$, where $\overline{\alpha} := \overline{\beta}f(1) - 1$.

Lemma 4.2. *Let $c > 0$ be given. Let $(\overline{\phi}, \overline{\psi})$, $(\underline{\phi}, \underline{\psi})$ be a pair of super and sub solutions of (4.4). Then, system (4.4) admits a unique solution (ϕ, ψ) such that*

$$(\underline{\phi}(\xi), \underline{\psi}(\xi)) \leq (\phi(\xi), \psi(\xi)) \leq (\overline{\phi}(\xi), \overline{\psi}(\xi)), \quad \xi \in \mathbb{R}.$$

Proof. For each $(\phi, \psi) \in X_h^2$, we define the nonlinear operators F_1, F_2 on X_h^2 by

$$F_1(\phi, \psi)(\xi) := M_1\phi(\xi) + d_1\widetilde{\mathcal{N}}_1[\phi](\xi) + f_1(\phi, \psi)(\xi),$$

$$F_2(\phi, \psi)(\xi) := M_1\psi(\xi) + d_2\widetilde{\mathcal{N}}_2[\psi](\xi) + f_2(\phi, \psi)(\xi),$$

where

$$f_1(\phi, \psi)(\xi) := -r_1\phi(\xi)[1 - \phi(\xi)] + f(1 - \phi(\xi))\psi(\xi),$$

$$f_2(\phi, \psi)(\xi) := r_2\psi(\xi)[\overline{\beta}f(1 - \phi(\xi)) - 1 - \psi(\xi)].$$

Here, M_1 denotes some large positive constant such that F_1 and F_2 satisfy proper monotonicity in ϕ and ψ, respectively. With these notions, let $P = (P_1, P_2) \colon X_h^2 \to X^2$ be

$$P_1(\phi, \psi)(\xi) := \frac{1}{c}\int_{-\infty}^{\xi} e^{-\frac{M_1(\xi - s)}{c}} F_1(\phi, \psi)(s)\, ds,$$

$$P_2(\phi, \psi)(\xi) := \frac{1}{c}\int_{-\infty}^{\xi} e^{-\frac{M_1(\xi - s)}{c}} F_2(\phi, \psi)(s)\, ds.$$

Let $\mu > 0$ be a small constant. Denote

$$B_\mu(\mathbb{R}, \mathbb{R}^2) := \left\{ (\phi, \psi) \in X^2 : |(\phi, \psi)|_\mu := \sup_{\xi \in \mathbb{R}} \max(|\phi(\xi)|, |\psi(\xi)|)e^{-\mu|\xi|} < \infty \right\}.$$

Then, $(B_\mu(\mathbb{R}, \mathbb{R}^2), |\cdot|_\mu)$ is a Banach space. Define

$$\Gamma := \{(\phi, \psi) \in X_h^2 : (\underline{\phi}, \underline{\psi}) \leq (\phi, \psi) \leq (\overline{\phi}, \overline{\psi})\}.$$

Thus, Γ is a nonempty convex, closed and bounded with respect to the norm $|\cdot|_\mu$.

Since $f_1(\phi, \psi)$ is monotone increasing in ψ and $f_2(\phi, \psi)$ is monotone decreasing in ϕ, we have

$$F_1(\phi, \psi)(\xi) \geq M_1\underline{\phi}(\xi) + d_1\widetilde{\mathcal{N}}_1[\underline{\phi}](\xi) + f_1(\underline{\phi}, \underline{\psi})(\xi) =: \underline{F}_1,$$
$$F_1(\phi, \psi)(\xi) \leq M_1\overline{\phi}(\xi) + d_1\widetilde{\mathcal{N}}_1[\overline{\phi}](\xi) + f_1(\overline{\phi}, \overline{\psi})(\xi) =: \overline{F}_1,$$
$$F_2(\phi, \psi)(\xi) \geq M_1\underline{\psi}(\xi) + d_2\widetilde{\mathcal{N}}_2[\underline{\psi}](\xi) + f_2(\overline{\phi}, \underline{\psi})(\xi) =: \underline{F}_2,$$
$$F_2(\phi, \psi)(\xi) \leq M_1\overline{\psi}(\xi) + d_2\widetilde{\mathcal{N}}_2[\overline{\psi}](\xi) + f_2(\underline{\phi}, \overline{\psi})(\xi) =: \overline{F}_2.$$

Let

$$\underline{P}_1(\xi) := \frac{1}{c}\int_{-\infty}^{\xi} e^{-\frac{M_1(\xi-s)}{c}}\underline{F}_1(s)\,ds, \quad \overline{P}_1(\xi) := \frac{1}{c}\int_{-\infty}^{\xi} e^{-\frac{M_1(\xi-s)}{c}}\overline{F}_1(s)\,ds, \quad \xi \in \mathbb{R}.$$

Then it suffices to prove that

$$\underline{\phi}(\xi) \leq \underline{P}_1(\xi) \leq \overline{P}_1(\xi) \leq \overline{\phi}(\xi), \quad \xi \in \mathbb{R}.$$

Indeed, let

$$E = \{E_1, E_2, \ldots, E_n\},$$

and denote $E_0 = -\infty$, $E_{n+1} = \infty$. If $\xi \in (E_{k-1}, E_k)$ with some $k \in \{1, 2, \ldots, n+1\}$, then

$$\begin{aligned}
\overline{P}_1(\xi) &= \frac{1}{c}\int_{-\infty}^{\xi} e^{-\frac{M_1(\xi-s)}{c}}\overline{F}_1(s)\,ds \\
&= \left(\sum_{i=0}^{k-1}\frac{1}{c}\int_{E_{i-1}}^{E_i} + \frac{1}{c}\int_{E_{i-1}}^{\xi}\right) e^{-\frac{M_1(\xi-s)}{c}}\overline{F}_1(s)\,ds \\
&\leq \left(\sum_{i=0}^{k-1}\frac{1}{c}\int_{E_{i-1}}^{E_i} + \frac{1}{c}\int_{E_{i-1}}^{\xi}\right) e^{-\frac{M_1(\xi-s)}{c}}(c\overline{\phi}'(s) + M_1\overline{\phi}(s))\,ds \\
&= \overline{\phi}(\xi).
\end{aligned}$$

Since $\overline{P}_1(\xi)$ and $\overline{\phi}(\xi)$ are continuous functions, we obtain

$$\overline{P}_1(\xi) \leq \overline{\phi}(\xi), \quad \xi \in \mathbb{R}.$$

In a similar way, we can deduce that

$$\underline{\phi}(\xi) \leq \underline{P}_1(\xi), \quad \xi \in \mathbb{R},$$

and

$$\underline{\psi}(\xi) \leq P_2(\phi, \psi)(\xi) \leq \overline{\psi}(\xi), \quad \xi \in \mathbb{R}.$$

Thus, we have $P(\Gamma) \subset \Gamma$. In addition, if μ satisfies $\mu < \frac{M_1}{2c}$ (the proof is similar to [16, Lemma 3.4]), we obtain that $P\colon \Gamma \to \Gamma$ is complete continuous in the sense of $|\cdot|_\mu$. Thus, using Schauder's fixed point theorem, we complete the proof of Lemma 4.2. $\square$

4.2. Existence of planar wave solutions for $c \geq c^*$

In order to construct a pair of proper super-sub solutions, we consider the function

$$\Delta(\lambda, c) := d_2 \left[\int_{\mathbb{R}^N} J_2(x) e^{\lambda x_1} \, \mathrm{d}x - 1 \right] - c\lambda + r_2(\overline{\beta} f(1) - 1)$$

$$= d_2 \left[\int_{\mathbb{R}} \widetilde{J}_2(x_1) e^{\lambda x_1} \, \mathrm{d}x_1 - 1 \right] - c\lambda + r_2(\overline{\beta} f(1) - 1),$$

where $\overline{\beta} := \frac{\beta}{r_2}$. By the convexity, we have the following conclusion.

Lemma 4.3. *Let Assumption* 1.1 *be satisfied. Then the following statements hold.*

(i) *If $c > c^*$, then the equation $\Delta(\lambda, c) = 0$ has two positive roots $\lambda_1(c)$, $\lambda_2(c)$, and $0 < \lambda_1(c) < \lambda_2(c) < \lambda_0$. Moreover, $\Delta(\lambda, c) < 0$, if $\lambda \in (\lambda_1(c), \lambda_2(c))$. $\Delta(\lambda, c) > 0$, if $\lambda \in (0, \lambda_1(c)) \cup (\lambda_2(c), \lambda_0)$.*

(ii) *If $0 < c < c^*$, then $\Delta(\lambda, c) > 0$ for all $\lambda \in (0, \lambda_0)$.*

(iii) *If $c = c^* > 0$, then $\Delta(\lambda, c^*) \geq 0$ for any $\lambda \in (0, \lambda_0)$ and $\Delta(\lambda, c^*) = 0$ has a unique root λ^*.*

The following theorem presents us the existence of nonnegative solutions of system (4.4) with $c > c^*$ connecting the predator-free equilibrium $(1, 0)$ at $\xi = -\infty$.

Theorem 4.4. *Let $c > c^*$ be given and fixed. Then, (4.4) admits a nonnegative solution (ϕ, ψ) such that $\limsup_{\xi \to -\infty} (\phi(\xi), \psi(\xi)) = (0, 0)$.*

Proof. Let $c > c^*$ be given and fixed. We define the following continuous functions:

$$\overline{\phi}(\xi) = \begin{cases} K e^{\lambda \xi}, & \xi \leq \xi_0, \\ 1, & \xi \geq \xi_0, \end{cases} \qquad \overline{\psi}(\xi) = \begin{cases} e^{\lambda_1 \xi}, & \xi \leq \xi_1, \\ \overline{\beta} f(1) - 1, & \xi \geq \xi_1, \end{cases}$$

$$\underline{\phi}(\xi) \equiv 0, \qquad \underline{\psi}(\xi) = \begin{cases} e^{\lambda_1 \xi} - q e^{\eta \lambda_1 \xi}, & \xi \leq \xi_2, \\ 0, & \xi \geq \xi_2. \end{cases}$$

Define

$$A(\lambda) := d_1 \left[\int_{\mathbb{R}} \widetilde{J}_1(x_1) e^{\lambda x_1} \, \mathrm{d}x_1 - 1 \right] - c\lambda.$$

Since $A(0) = 0$ and

$$A'(0) = \lim_{\lambda \to 0} \left\{ d_1 \int_{\mathbb{R}} \widetilde{J}_1(x_1) x_1 e^{\lambda x_1} \, \mathrm{d}x_1 - c \right\} = -c < 0,$$

there exists $\lambda \in (0, \min\{\lambda_0, \lambda_1(c)\})$ such that

(4.9) $$A(\lambda) = d_1 \left[\int_{\mathbb{R}} \widetilde{J}_1(x_1) e^{\lambda x_1} \, \mathrm{d}x_1 - 1 \right] - c\lambda < 0.$$

For notational simplicity, since c is fixed in this proof, we simply denote $\lambda_i = \lambda_i(c)$, $i = 1, 2$.

First, we define ξ_1 by $e^{\lambda_1 \xi_1} = \overline{\beta} f(1) - 1$ and choose a constant $\xi_0 < \min\{0, \xi_1\}$ such that $\overline{a} e^{(\lambda_1 - \lambda)\xi_0} < 1$. Next, we set $K = e^{-\lambda \xi_0}$. Clearly $K > 1$ and

$$(4.10) \qquad \overline{a} e^{\lambda_1 \xi} < e^{\lambda \xi}, \quad \forall \xi < \xi_0.$$

Then, take $\eta \in (1, 2)$ such that

$$(4.11) \qquad \eta \lambda_1 < \min\{\lambda_2, \lambda_1 + \lambda\}.$$

Finally, for $q > 1$ define $\xi_2 = \xi_2(q) < 0$ by $e^{(\eta-1)\lambda_1 \xi_2} = \frac{1}{q}$. Letting $q \to \infty$, we have $\xi_2 \to -\infty$. We can take a constant $q > 1$ large enough such that $\xi_2 \leq \xi_0$ and

$$(4.12) \qquad q > \frac{r_2 + r_2 \overline{\beta} f(1) K}{-\triangle(\eta \lambda_1, c)} + 1.$$

Obviously, $\underline{\phi}(\xi) \leq \overline{\phi}(\xi)$ for any $\xi \in \mathbb{R}$. Since $\xi_2 < \xi_1$, we also have $\underline{\psi}(\xi) \leq \overline{\psi}(\xi)$ for all $\xi \in \mathbb{R}$.

Now we prove that the functions $(\overline{\phi}, \overline{\psi})$ and $(\underline{\phi}, \underline{\psi})$ are a pair of super and sub solutions of (4.4) with $(\overline{\phi}, \overline{\psi})(-\infty) = (\underline{\phi}, \underline{\psi})(-\infty) = (0, 0)$.

For $\xi > \xi_0$, we have $\overline{\phi}(\xi) = 1$ and $\overline{\phi}'(\xi) = 0$, so that

$$d_1 \widetilde{\mathcal{N}}_1[\overline{\phi}](\xi) - r_1 \overline{\phi}(\xi)[1 - \overline{\phi}(\xi)] + f(1 - \overline{\phi}(\xi))\overline{\psi}(\xi) - c\overline{\phi}'(\xi)$$
$$= d_1 \left[\int_{\mathbb{R}} \widetilde{J}_1(x_1)\overline{\phi}(\xi - x_1)\, dx_1 - \overline{\phi}(\xi) \right] \leq 0.$$

Thus, condition (4.5) holds for $\xi > \xi_0$.

For $\xi < \xi_0$, we have $\overline{\phi}(\xi) = K e^{\lambda \xi} < 1$. Moreover, $\overline{\phi}(\xi) \leq K e^{\lambda \xi}$ for all $\xi \in \mathbb{R}$ and $\overline{\psi}(\xi) = e^{\lambda_1 \xi}$ for all $\xi < \xi_0 \leq \xi_1$. Using $f(x) < ax$ and the symmetry of the function $\widetilde{J}_1$, it follows from (4.9) and (4.10) that

$$d_1 \widetilde{\mathcal{N}}_1[\overline{\phi}](\xi) + f(1 - \overline{\phi}(\xi))\overline{\psi}(\xi) - r_1 \overline{\phi}(\xi)[1 - \overline{\phi}(\xi)] - c\overline{\phi}'(\xi)$$
$$\leq d_1 \left[\int_{\mathbb{R}} \widetilde{J}_1(x_1)\overline{\phi}(\xi - x_1)\, dx_1 - K e^{\lambda \xi} \right] + a(1 - \overline{\phi}(\xi))\overline{\psi}(\xi) - r_1 \overline{\phi}(\xi)[1 - K e^{\lambda \xi}] - c\lambda K e^{\lambda \xi}$$
$$\leq d_1 \left[\int_{\mathbb{R}} K\widetilde{J}_1(x_1)e^{\lambda(\xi - x_1)}\, dx_1 - K e^{\lambda \xi} \right] + r_1(1 - K e^{\lambda \xi})\left[\frac{a}{r_1} e^{\lambda_1 \xi} - K e^{\lambda \xi} \right] - c\lambda K e^{\lambda \xi}$$
$$\leq K e^{\lambda \xi} \left\{ d_1 \left[\int_{\mathbb{R}} \widetilde{J}_1(x_1)e^{\lambda x_1}\, dx_1 - 1 \right] - c\lambda \right\} \leq 0.$$

Hence, condition (4.5) also holds for $\xi < \xi_0$ and thus for any $\xi \neq \xi_0$.

We now consider (4.6). For $\xi > \xi_1$, we get $\overline{\psi}(\xi) = \overline{\beta} f(1) - 1$. Since $\overline{\psi}(\xi) \leq \overline{\beta} f(1) - 1$ for all $\xi \in \mathbb{R}$, it follows that

$$d_2 \widetilde{\mathcal{N}}_2[\overline{\psi}](\xi) + r_2 \overline{\psi}(\xi)[\overline{\beta} f(1 - \underline{\phi}(\xi)) - 1 - \overline{\psi}(\xi)] - c\overline{\psi}'(\xi)$$

$$\leq d_2\left[\int_{\mathbb{R}}(\bar{\beta}f(1)-1)\widetilde{J}_2(x_1)\,\mathrm{d}x_1-(\bar{\beta}f(1)-1)\right]+r_2(\bar{\beta}f(1)-1)[\bar{\beta}f(1)-1-(\bar{\beta}f(1)-1)]$$

$$=0.$$

Hence, (4.6) holds for $\xi>\xi_1$.

For $\xi<\xi_1$, we get $\overline{\psi}(\xi)=\mathrm{e}^{\lambda_1\xi}$. Applying $\overline{\psi}(\xi)\leq\mathrm{e}^{\lambda_1\xi}$ for all $\xi\in\mathbb{R}$ and $\underline{\phi}(\xi)\equiv 0$, we deduce that

$$d_2\widetilde{\mathcal{N}}_2[\overline{\psi}](\xi)-c\overline{\psi}'(\xi)+r_2\overline{\psi}(\xi)[\bar{\beta}f(1-\underline{\phi}(\xi))-1-\overline{\psi}(\xi)]$$
$$=d_2\widetilde{\mathcal{N}}_2[\overline{\psi}](\xi)-c\overline{\psi}'(\xi)+r_2\overline{\psi}(\xi)[\bar{\beta}f(1)-1-\overline{\psi}(\xi)]$$
$$\leq d_2\widetilde{\mathcal{N}}_2[\overline{\psi}](\xi)-c\overline{\psi}'(\xi)+r_2(\bar{\beta}f(1)-1)\overline{\psi}(\xi)$$
$$=d_2\left[\int_{\mathbb{R}}\overline{\psi}(\xi-x_1)\widetilde{J}_2(x_1)\,\mathrm{d}x_1-\mathrm{e}^{\lambda_1\xi}\right]-c\lambda_1\mathrm{e}^{\lambda_1\xi}+r_2(\bar{\beta}f(1)-1)\mathrm{e}^{\lambda_1\xi}$$
$$\leq\mathrm{e}^{\lambda_1\xi}\triangle(\lambda_1,c)=0$$

for all $\xi<\xi_1$. Hence, (4.6) also holds for $\xi<\xi_1$ and thus for all $\xi\in\mathbb{R}\setminus\{\xi_1\}$.

Now consider (4.7). Since $\underline{\phi}(\xi)\equiv 0$, using (H1) and $\underline{\psi}(\xi)\geq 0$, it follows that

$$d_1\widetilde{\mathcal{N}}_1[\underline{\phi}](\xi)-r_1\underline{\phi}(\xi)[1-\underline{\phi}(\xi)]+f(1-\underline{\phi}(\xi))\underline{\psi}(\xi)-c\underline{\phi}'(\xi)=f(1)\underline{\psi}(\xi)\geq 0.$$

Hence, (4.7) holds for all $\xi\in\mathbb{R}$. It remains to check that (4.8) holds true. Clearly, for $\xi>\xi_2$, we have $\underline{\psi}(\xi)=0$ and (4.8) holds true for $\xi>\xi_2$.

For $\xi<\xi_2$, one has $\underline{\psi}(\xi)=\mathrm{e}^{\lambda_1\xi}-q\mathrm{e}^{\eta\lambda_1\xi}$. Since $\xi_2\leq\xi_0$, we have $\overline{\phi}(\xi)=K\mathrm{e}^{\lambda\xi}$, thus $0<\overline{\phi}(\xi)<1$. In addition, for any $\xi\in\mathbb{R}$, one has $\underline{\psi}(\xi)\leq\mathrm{e}^{\lambda_1\xi}$ and $\underline{\psi}(\xi)\geq\mathrm{e}^{\lambda_1\xi}-q\mathrm{e}^{\eta\lambda_1\xi}$. Using (H3), $\xi<\xi_2<0$ and $\triangle(\lambda_1,c)=0$, from (4.11) and (4.12), it follows that

$$d_2\widetilde{\mathcal{N}}_2[\underline{\psi}](\xi)+r_2\underline{\psi}(\xi)[\bar{\beta}f(1-\overline{\phi}(\xi))-1-\underline{\psi}(\xi)]-c\underline{\psi}'(\xi)$$
$$\geq d_2\left[\int_{\mathbb{R}}\widetilde{J}_2(x_1)\underline{\psi}(\xi-x_1)\,\mathrm{d}x_1-(\mathrm{e}^{\lambda_1\xi}-q\mathrm{e}^{\eta\lambda_1\xi})\right]$$
$$\quad+r_2\underline{\psi}(\xi)[\bar{\beta}f(1)(1-\overline{\phi}(\xi))-1-\underline{\psi}(\xi)]-c(\lambda_1\mathrm{e}^{\lambda_1\xi}-q\lambda_1\eta\mathrm{e}^{\eta\lambda_1\xi})$$
$$=d_2\left[\int_{\mathbb{R}}\widetilde{J}_2(x_1)\underline{\psi}(\xi-x_1)\,\mathrm{d}x_1-(\mathrm{e}^{\lambda_1\xi}-q\mathrm{e}^{\eta\lambda_1\xi})\right]$$
$$\quad+r_2\underline{\psi}(\xi)(\bar{\beta}f(1)-1)-r_2\bar{\beta}f(1)\underline{\psi}(\xi)\overline{\phi}(\xi)-r_2\underline{\psi}^2(\xi)-c(\lambda_1\mathrm{e}^{\lambda_1\xi}-q\lambda_1\eta\mathrm{e}^{\eta\lambda_1\xi})$$
$$\geq d_2\left[\int_{\mathbb{R}}\widetilde{J}_2(x_1)(\mathrm{e}^{\lambda_1(\xi-x_1)}-q\mathrm{e}^{\eta\lambda_1(\xi-x_1)})\,\mathrm{d}x_1-(\mathrm{e}^{\lambda_1\xi}-q\mathrm{e}^{\eta\lambda_1\xi})\right]$$
$$\quad+r_2(\bar{\beta}f(1)-1)(\mathrm{e}^{\lambda_1\xi}-q\mathrm{e}^{\eta\lambda_1\xi})-r_2\bar{\beta}f(1)K\mathrm{e}^{(\lambda+\lambda_1)\xi}-r_2\mathrm{e}^{2\lambda_1\xi}-c(\lambda_1\mathrm{e}^{\lambda_1\xi}-q\lambda_1\eta\mathrm{e}^{\eta\lambda_1\xi})$$
$$\geq\mathrm{e}^{\lambda_1\xi}\triangle(\lambda_1,c)+\mathrm{e}^{\eta\lambda_1\xi}\left\{-q\triangle(\eta\lambda_1,c)-r_2\mathrm{e}^{(2-\eta)\lambda_1\xi}-r_2\bar{\beta}f(1)K\mathrm{e}^{(\lambda+\lambda_1-\eta\lambda_1)\xi}\right\}$$
$$\geq\mathrm{e}^{\eta\lambda_1\xi}[-q\triangle(\eta\lambda_1,c)-r_2-r_2\bar{\beta}f(1)K]>0.$$

Then we get that (4.8) also holds for $\xi < \xi_2$ and thus for any $\xi \in \mathbb{R} \setminus \{\xi_2\}$.

It follows from Lemma 4.2 that there exists a unique solution (ϕ, ψ) with $(\underline{\phi}, \underline{\psi}) \leq (\phi, \psi) \leq (\overline{\phi}, \overline{\psi})$. Since $(\overline{\phi}, \overline{\psi})(-\infty) = (\underline{\phi}, \underline{\psi})(-\infty) = (0, 0)$, we conclude that $(\phi, \psi)(-\infty) = (0, 0)$. This is the end. $\square$

We now derive the existence of nonnegative solutions of system (4.4) with $c = c^*$ connecting the predator-free equilibrium $(1, 0)$ at $\xi = -\infty$.

Theorem 4.5. *Assume that the function J_2 is compactly supported. Then, system* (4.4) *with $c = c^*$ admits a nonnegative solution (ϕ, ψ) such that $\lim_{\xi \to -\infty}(\phi(\xi), \psi(\xi)) = (0, 0)$.*

Proof. The approach is to construct a suitable pair of super and sub solutions satisfying the limit condition $(\overline{\phi}, \overline{\psi})(-\infty) = (\underline{\phi}, \underline{\psi})(-\infty) = (0, 0)$.

We set the following functions:

$$\overline{\phi}(\xi) = \begin{cases} K e^{\lambda \xi}, & \xi \leq \xi_0, \\ 1, & \xi \geq \xi_0, \end{cases} \quad \overline{\psi}(\xi) = \begin{cases} -L \xi e^{\lambda^* \xi}, & \xi \leq \xi_1, \\ \overline{\beta} f(1) - 1, & \xi \geq \xi_1, \end{cases}$$

$$\underline{\phi}(\xi) \equiv 0, \quad \underline{\psi}(\xi) = \begin{cases} (-L\xi - q\sqrt{-\xi}) e^{\lambda^* \xi}, & \xi \leq \xi_3, \\ 0, & \xi \geq \xi_3, \end{cases}$$

where L, K, q are constants to be determined (in order) later while the constant $\lambda \in (0, \min\{\lambda_0, \lambda^*/2\})$ is defined as before so that (4.9) holds with $c = c^*$.

Since J_2 has a compact support, we choose a positive constant G such that

$$J_2(x) = 0, \quad \|x\| > G.$$

For any large enough constant L, the following equation $-L\xi e^{\lambda^* \xi} = \overline{\beta} f(1) - 1$ has two negative roots, denoted by ξ_1, ξ_2 and $\xi_1 < \xi_2$, and

(4.13) $\xi_2 - \xi_1 > G.$

It is easy to see that

$$\xi_1 < -\frac{1}{\lambda^*} < \xi_2 < 0 \quad \text{and} \quad L\xi e^{\lambda^* \xi} > \overline{\beta} f(1) - 1 \quad \text{for all } \xi \in (\xi_1, \xi_2).$$

Fix $\lambda_* \in (\lambda^* - \lambda, \lambda^*)$. Since $\lambda < \lambda^*/2$, we have $\lambda_* > \lambda$. Because $\lambda_* - \lambda^* - \lambda < 0$, we can take $\xi_0 < 0$ such that

(4.14) $\xi_0 < \min\{-1/(\lambda^* - \lambda_*), \xi_1\}, \quad e^{(\lambda_* - \lambda^* - \lambda)\xi_0}/(-\xi_0) > \overline{a} L.$

For such a fixed ξ_0, we set $K := e^{-\lambda \xi_0}$. Define function $s \colon \mathbb{R} \to \mathbb{R}$ as $s(\xi) := (-\xi) e^{(\lambda^* - \lambda_*)\xi}$. It easy to see that $s(\xi)$ is increasing on $(-\infty, -1/(\lambda^* - \lambda_*))$. Then, we have

$$s(\xi) < s(\xi_0) \quad \text{for } \xi \leq \xi_0,$$

hence, it follows that

$$-\overline{a}L\xi e^{(\lambda^*-\lambda_*)\xi} < -\overline{a}L\xi_0 e^{(\lambda^*-\lambda_*)\xi_0}.$$

Since $\lambda_* > \lambda$ and (4.14), we deduce that

$$(4.15) \qquad -\overline{a}L\xi e^{\lambda^*\xi} < Ke^{\lambda_*\xi} < Ke^{\lambda\xi} \quad \text{for all } \xi \le \xi_0.$$

According to [2,6], we know that the function $z \mapsto Y(z) := [-Lz - q(-z)^{1/2}]e^{\lambda^*z}$ is positive and has a unique critical (maximal) point in $(-\infty, \xi_3)$, where $\xi_3 := -(q/L)^2$, for any $q > 0$. Then, we take q large enough such that

$$(4.16) \qquad \xi_3 < \xi_0,$$

$$(4.17) \qquad q \ge \frac{\max_{\xi<0}\left\{8(-\xi+G)^{3/2}\left[r_2\overline{\beta}f(1)\frac{K^2}{\overline{a}}e^{(\lambda+\lambda_*-\lambda^*)\xi} + r_2\frac{K^2}{\overline{a}^2}e^{(2\lambda_*-\lambda^*)\xi}\right]\right\}}{d_2\int_{\mathbb{R}}\widetilde{J}_2(x_1)x_1^2 e^{-\lambda^*x_1}\,\mathrm{d}x_1}.$$

It is clear that $\underline{\phi} \le \overline{\phi}$ on $\mathbb{R}$, and $\underline{\psi} < \overline{\psi}$ on $\mathbb{R}$ by (4.16), $\xi_3 < \xi_0 < \xi_1$. Clearly (4.5) holds for $\xi > \xi_0$.

For $\xi < \xi_0$, one has $\overline{\phi}(\xi) = Ke^{\lambda\xi} < 1$. Since $f(x) \le ax$ and $\overline{\phi}(\xi) \le Ke^{\lambda\xi}$ for all $\xi \in \mathbb{R}$. Using $\xi_0 < \xi_1$, (4.9) and (4.15), it follows that

$$d_1\widetilde{\mathcal{N}}_1[\overline{\phi}](\xi) + f(1-\overline{\phi}(\xi))\overline{\psi}(\xi) - r_1\overline{\phi}(\xi)[1-\overline{\phi}(\xi)] - c^*\overline{\phi}'(\xi)$$

$$\le d_1\left[\int_{\mathbb{R}}\widetilde{J}_1(x_1)\overline{\phi}(\xi-x_1)\,\mathrm{d}x_1 - Ke^{\lambda\xi}\right] + a(1-\overline{\phi}(\xi))\overline{\psi}(\xi) - r_1\overline{\phi}(\xi)(1-Ke^{\lambda\xi}) - c^*\lambda Ke^{\lambda\xi}$$

$$\le Ke^{\lambda\xi}\left\{d_1\left[\int_{\mathbb{R}}\widetilde{J}_1(x_1)e^{\lambda x_1}\,\mathrm{d}x_1 - 1\right] - c^*\lambda\right\} + r_1(1-Ke^{\lambda\xi})\left[\frac{-a}{r_1}L\xi e^{\lambda^*\xi} - Ke^{\lambda\xi}\right] < 0$$

for $\xi < \xi_0$. Hence, condition (4.5) also holds for $\xi < \xi_0$ and thus for any $\xi \ne \xi_0$.

We now turn to (4.6). For $\xi > \xi_1$, we have $\overline{\psi}(\xi) = \overline{\beta}f(1) - 1$. Since $\overline{\psi}(\xi) \le \overline{\beta}f(1) - 1$ for all $\xi \in \mathbb{R}$, we have

$$d_2\widetilde{\mathcal{N}}_2[\overline{\psi}](\xi) + r_2\overline{\psi}(\xi)[\overline{\beta}f(1-\underline{\phi}(\xi)) - 1 - \overline{\psi}(\xi)] - c\overline{\psi}'(\xi)$$

$$\le d_2\left[\int_{\mathbb{R}}(\overline{\beta}f(1)-1)\widetilde{J}_2(x_1)\,\mathrm{d}x_1 - (\overline{\beta}f(1)-1)\right]$$

$$+ r_2(\overline{\beta}f(1)-1)[\overline{\beta}f(1)-1-(\overline{\beta}f(1)-1)] = 0.$$

Hence, (4.6) holds for $\xi > \xi_1$.

For $\xi < \xi_1$, from the assumption that J_2 has a compact support in $\overline{B}_G(0)$ and (4.13), we obtain $\xi - x_1 \le \xi_1 + G < \xi_2$ for all $x_1 \in [-G, G]$ and

$$\int_{\mathbb{R}}\overline{\psi}(\xi-x_1)\widetilde{J}_2(x_1)\,\mathrm{d}x_1 = \int_{-G}^{G}\overline{\psi}(\xi-x_1)\widetilde{J}_2(x_1)\,\mathrm{d}x_1$$

$$\le \int_{\mathbb{R}}\left\{-L(\xi-x_1)e^{\lambda^*(\xi-x_1)}\right\}\widetilde{J}_2(x_1)\,\mathrm{d}x_1.$$

Hence, we have

$$
\begin{aligned}
&d_2\widetilde{\mathcal{N}}_2[\overline{\psi}](\xi) - c^*\overline{\psi}'(\xi) + r_2\overline{\psi}(\xi)[\overline{\beta}f(1-\underline{\phi}(\xi)) - 1 - \overline{\psi}(\xi)]\\
&= d_2\widetilde{\mathcal{N}}_2[\overline{\psi}](\xi) - c^*\overline{\psi}'(\xi) + r_2\overline{\psi}(\xi)[\overline{\beta}f(1) - 1 - \overline{\psi}(\xi)]\\
&\le d_2\widetilde{\mathcal{N}}_2[\overline{\psi}](\xi) - c^*\overline{\psi}'(\xi) + r_2(\overline{\beta}f(1) - 1)\overline{\psi}(\xi)\\
&= d_2\left[\int_{\mathbb{R}}\overline{\psi}(\xi - x_1)\widetilde{J}_2(x_1)\,\mathrm{d}x_1 + L\xi e^{\lambda^*\xi}\right] + c^*L(\lambda^*\xi + 1)e^{\lambda^*\xi} - r_2L(\overline{\beta}f(1) - 1)\xi e^{\lambda^*\xi}\\
&\le d_2\left[\int_{\mathbb{R}}\{-L(\xi - x_1)e^{\lambda^*(\xi - x_1)}\}\widetilde{J}_2(x_1)\,\mathrm{d}x_1 + L\xi e^{\lambda^*\xi}\right]\\
&\quad + c^*L(\lambda^*\xi + 1)e^{\lambda^*\xi} - r_2L(\overline{\beta}f(1) - 1)\xi e^{\lambda^*\xi}\\
&= -L\xi e^{\lambda^*\xi}\left\{d_2\left[\int_{\mathbb{R}}e^{\lambda^*x_1}\widetilde{J}_2(x_1)\,\mathrm{d}x_1 - 1\right] - c^*\lambda^* + r_2(\overline{\beta}f(1) - 1)\right\}\\
&\quad + Le^{\lambda^*\xi}\left[d_2\int_{\mathbb{R}}x_1e^{-\lambda^*x_1}\widetilde{J}_2(x_1)\,\mathrm{d}x_1 + c^*\right] = 0
\end{aligned}
$$

for all $\xi < \xi_1$, where we use a change of variable $z = -x_1$ yielding

$$
(4.18)\qquad d_2\int_{-\infty}^{+\infty}x_1e^{-\lambda^*x_1}\widetilde{J}_2(x_1)\,dx_1 = -d_2\int_{-\infty}^{+\infty}ze^{\lambda^*z}\widetilde{J}_2(z)\,dz = -c^*.
$$

Hence, (4.6) also holds for $\xi < \xi_1$ and thus for all $\xi \in \mathbb{R}\setminus\{\xi_1\}$.

Now consider (4.7). Since $\underline{\phi}(\xi) \equiv 0$, using (H2), and $\underline{\psi}(\xi) \ge 0$, we deduce that

$$
d_1\widetilde{\mathcal{N}}_1[\underline{\phi}](\xi) - r_1\underline{\phi}(\xi)[1 - \underline{\phi}(\xi)] + f(1 - \underline{\phi}(\xi))\underline{\psi}(\xi) - c\underline{\phi}'(\xi) = f(1)\underline{\psi}(\xi) \ge 0.
$$

Hence, for all $\xi \in \mathbb{R}$ (4.7) holds. It remains to check that (4.8) holds true. To do so, observe that for $\xi > \xi_3$, we have $\underline{\psi}(\xi) = 0$ and (4.8) holds true for $\xi > \xi_3$.

For $\xi < \xi_3$, one has $\underline{\psi}(\xi) = (-L\xi - q\sqrt{-\xi})e^{\lambda^*\xi}$. Furthermore, for all $z \in \mathbb{R}$, we have $\underline{\psi}(z) \ge (-Lz - q\sqrt{-z})e^{\lambda^*z}$. Using (H3), (4.15), (4.16) and

$$
d_2\left[\int_{\mathbb{R}}\widetilde{J}_2(x_1)e^{-\lambda^*x_1}\,\mathrm{d}x_1 - 1\right] - c^*\lambda^* + r_2(\overline{\beta}f(1) - 1) = 0, \quad \int_{\mathbb{R}}\widetilde{J}_2(x_1)x_1e^{-\lambda^*x_1}\,\mathrm{d}x_1 + c^* = 0,
$$

it follows that

$$
\begin{aligned}
&d_2\widetilde{\mathcal{N}}_2[\underline{\psi}](\xi) - c^*\underline{\psi}'(\xi) + r_2\underline{\psi}(\xi)[\overline{\beta}f(1-\overline{\phi}(\xi)) - 1 - \underline{\psi}(\xi)]\\
&\ge d_2\left[\int_{\mathbb{R}}\widetilde{J}_2(x_1)\underline{\psi}(\xi - x_1)\,\mathrm{d}x_1 - \underline{\psi}(\xi)\right] - c^*\underline{\psi}'(\xi) + r_2\underline{\psi}(\xi)[\overline{\beta}f(1)(1 - \overline{\phi}(\xi)) - 1 - \underline{\psi}(\xi)]\\
&\ge d_2\int_{\mathbb{R}}\widetilde{J}_2(x_1)\left\{[-L(\xi - x_1) - q\sqrt{-(\xi - x_1)}]e^{\lambda^*(\xi - x_1)}\right\}\,\mathrm{d}x_1 - d_2(-L\xi - q\sqrt{-\xi})e^{\lambda^*\xi}\\
&\quad + c^*L(1 + \lambda^*\xi)e^{\lambda^*\xi} + c^*q\left(\lambda^*\sqrt{-\xi} - \frac{1}{2\sqrt{-\xi}}\right)e^{\lambda^*\xi}\\
&\quad + r_2\underline{\psi}(\xi)(\overline{\beta}f(1) - 1) - r_2\overline{\beta}f(1)\overline{\psi}(\xi)\overline{\phi}(\xi) - r_2\overline{\psi}^2(\xi)
\end{aligned}
$$

$$\geq d_2 \mathrm{e}^{\lambda^* \xi} \left[\int_{\mathbb{R}} \widetilde{J}_2(x_1)[-L(\xi - x_1)]\mathrm{e}^{-\lambda^* x_1}\, \mathrm{d}x_1 + L\xi \right] + c^* L(1 + \lambda^* \xi)\mathrm{e}^{\lambda^* \xi}$$

$$- q d_2 \mathrm{e}^{\lambda^* \xi} \left[\int_{\mathbb{R}} \widetilde{J}_2(x_1)\sqrt{-(\xi - x_1)}\mathrm{e}^{-\lambda^* x_1}\, \mathrm{d}x_1 - \sqrt{-\xi} \right] + c^* q \left(\lambda^* \sqrt{-\xi} - \frac{1}{2\sqrt{-\xi}} \right) \mathrm{e}^{\lambda^* \xi}$$

$$+ r_2(\overline{\beta} f(1) - 1)[(-L\xi - q\sqrt{-\xi})\mathrm{e}^{\lambda^* \xi}] - r_2 \overline{\beta} f(1) \frac{K^2}{\overline{a}} \mathrm{e}^{(\lambda + \lambda_*)\xi} - r_2 \frac{K^2}{\overline{a}^2} \mathrm{e}^{2\lambda_* \xi}$$

$$= -q d_2 \mathrm{e}^{\lambda^* \xi} \left[\int_{\mathbb{R}} \widetilde{J}_2(x_1)\sqrt{-(\xi - x_1)}\mathrm{e}^{-\lambda^* x_1}\, \mathrm{d}x_1 - \sqrt{-\xi} \right] + c^* q \left(\lambda^* \sqrt{-\xi} - \frac{1}{2\sqrt{-\xi}} \right) \mathrm{e}^{\lambda^* \xi}$$

$$- r_2(\overline{\beta} f(1) - 1)q\sqrt{-\xi}\mathrm{e}^{\lambda^* \xi} - r_2 \overline{\beta} f(1) \frac{K^2}{\overline{a}} \mathrm{e}^{(\lambda + \lambda_*)\xi} - r_2 \frac{K^2}{\overline{a}^2} \mathrm{e}^{2\lambda_* \xi}$$

$$:= \mathrm{e}^{\lambda^* \xi}[q I_1(\xi) - I_2(\xi)],$$

where

$$I_1(\xi) := -d_2 \left[\int_{\mathbb{R}} \widetilde{J}_2(x_1)\sqrt{-(\xi - x_1)}\mathrm{e}^{-\lambda^* x_1}\, \mathrm{d}x_1 - \sqrt{-\xi} \right]$$

$$+ c^* \left(\lambda^* \sqrt{-\xi} - \frac{1}{2\sqrt{-\xi}} \right) - r_2(\overline{\beta} f(1) - 1)\sqrt{-\xi},$$

$$I_2(\xi) := r_2 \overline{\beta} f(1) \frac{K^2}{\overline{a}} \mathrm{e}^{(\lambda + \lambda_* - \lambda^*)\xi} + r_2 \frac{K^2}{\overline{a}^2} \mathrm{e}^{(2\lambda_* - \lambda^*)\xi}.$$

Next, to get the conclusion, we need to deduce a lower bound estimate of I_1 for $\xi < \xi_3$. Using $\triangle(\lambda^*, c^*) = 0$ and (4.18), it follows that

$$I_1(\xi) = -d_2 \left\{ \int_{\mathbb{R}} \widetilde{J}_2(x_1)\left[\sqrt{-\xi} + \sqrt{-(\xi - x_1)} - \sqrt{-\xi} \right]\mathrm{e}^{-\lambda^* x_1}\, \mathrm{d}x_1 - \sqrt{-\xi} \right\}$$

$$+ c^* \left(\lambda^* \sqrt{-\xi} - \frac{1}{2\sqrt{-\xi}} \right) - r_2(\overline{\beta} f(1) - 1)\sqrt{-\xi}$$

$$= -d_2 \left\{ \int_{\mathbb{R}} \widetilde{J}_2(x_1)\left[\sqrt{-(\xi - x_1)} - \sqrt{-\xi} \right]\mathrm{e}^{-\lambda^* x_1}\, \mathrm{d}x_1 \right\} - \frac{c^*}{2\sqrt{-\xi}}$$

$$= d_2 \left\{ \int_{\mathbb{R}} \widetilde{J}_2(x_1)\left[\sqrt{-\xi} - \sqrt{-(\xi - x_1)} \right]\mathrm{e}^{-\lambda^* x_1}\, \mathrm{d}x_1 \right\} + \frac{d_2}{2\sqrt{-\xi}} \int_{\mathbb{R}} \widetilde{J}_2(x_1)x_1 \mathrm{e}^{-\lambda^* x_1}\, \mathrm{d}x_1$$

$$:= d_2 \int_{\mathbb{R}} \widetilde{J}_2(x_1)Q(\xi, x_1)\mathrm{e}^{-\lambda^* x_1}\, \mathrm{d}x_1,$$

where $Q(\xi, x_1)$ is defined by

$$Q(\xi, x_1) := \frac{x_1}{2\sqrt{-\xi}} + \sqrt{-\xi} - \sqrt{-(\xi - x_1)}.$$

Clearly

$$Q(\xi, x_1) = \frac{x_1}{2\sqrt{-\xi}} - \frac{x_1}{\sqrt{-\xi} - \sqrt{-(\xi - x_1)}} = \frac{x_1\left[\sqrt{-(\xi - x_1)} - \sqrt{-\xi} \right]}{2\sqrt{-\xi}\left[\sqrt{-(\xi - x_1)} + \sqrt{-\xi} \right]}$$

$$= \frac{x_1^2}{2\sqrt{-\xi}\left[\sqrt{-(\xi - x_1)} + \sqrt{-\xi} \right]^2}.$$

Because $J_2 = 0$ outside $\overline{B}_G(0)$ and

$$2\sqrt{-\xi}\left[\sqrt{-(\xi - x_1)} + \sqrt{-\xi}\right]^2 \le 8(-\xi + G)^{3/2} \quad \text{for } |x_1| < G,$$

we get the following lower estimate

(4.19) $$I_1(\xi) \ge \frac{d_2}{8(-\xi + G)^{3/2}} \int_{\mathbb{R}} \widetilde{J}_2(x_1) x_1^2 e^{-\lambda^* x_1} \, dx_1 \quad \text{for } \xi < \xi_3.$$

Finally, we use (4.17) and (4.19) to infer that

$$d_2 \widetilde{\mathcal{N}}_2[\underline{\psi}](\xi) - c^* \underline{\psi}'(\xi) + r_2 \underline{\psi}(\xi)[\overline{\beta} f(1 - \overline{\phi}(\xi)) - 1 - \underline{\psi}(\xi)] \ge 0$$

for all $\xi < \xi_3$. This completes the proof of the theorem. $\square$

The following theorem gives the proof of the existence of nonnegative solutions of system (4.4) with $c \ge c^*$ connecting the nontrivial state at $\xi = \infty$.

Theorem 4.6. *Let $c \ge c^*$ and let (ϕ, ψ) be any solution provided by Theorems 4.4 or 4.5. Then, it holds that $\limsup_{\xi \to \infty} \phi(\xi) < 1$. Moreover, if we assume that $\overline{a}\overline{\beta} f(1) < 1$, $\liminf_{\xi \to \infty} \psi(\xi) > 0$ also hold true.*

Proof. Let

$$(U, V)(x, t) := (1 - \phi, \psi)(x \cdot \nu + ct).$$

Since $V \le \overline{\beta} f(1) - 1$ in $\mathbb{R}^N \times \mathbb{R}$ and (H2), it follows from (1.1) that $U = U(x, t)$ satisfies

$$\frac{\partial U(x, t)}{\partial t} \ge d_1 \mathcal{N}_1[U(\cdot, t)](x) + r_1 U(x, t)\{[1 - \overline{a}(\overline{\beta} f(1) - 1)] - U(x, t)\}, \quad x \in \mathbb{R}^N, \ t > 0,$$

such that $U(x, 0) = 1 - \phi(x \cdot \nu) \ge 0$ and $U \not\equiv 0$. Indeed, letting $x \cdot \nu \to -\infty$, we have $U(x, 0) \to 1$. Applying Proposition 2.1, we obtain that $U \ge z$, where z is the solution to

$$\begin{cases} \frac{\partial z(x, t)}{\partial t} = d_1 \mathcal{N}_1[z(\cdot, t)](x) + r_1 z(x, t)\{[1 - \overline{a}(\overline{\beta} f(1) - 1)] - z(x, t)\}, & x \in \mathbb{R}^N, \ t > 0, \\ z(x, 0) = U(x, 0), & x \in \mathbb{R}^N. \end{cases}$$

Also, from Proposition 2.3(ii), it follows that $z(x, t) \to 1 - \overline{a}(\overline{\beta} f(1) - 1)$ as $t \to \infty$ for all $x \in \mathbb{R}^N$. Hence, we obtain that

$$\liminf_{t \to \infty} U(0, t) \ge \liminf_{t \to \infty} z(0, t),$$

i.e.,

$$\limsup_{t \to \infty} \phi(ct) = 1 - \liminf_{t \to \infty} U(0, t) \le 1 - \liminf_{t \to \infty} z(0, t) = \overline{a}(\overline{\beta} f(1) - 1).$$

Since $c > 0$, we conclude that

(4.20) $$\limsup_{\xi \to \infty} \phi(\xi) \le \overline{a}(\overline{\beta} f(1) - 1) < 1.$$

To prove the second part, we first show a claim:

$$(4.21) \qquad\qquad D := \sup_{\xi \in \mathbb{R}} \phi(\xi) \leq \overline{a}(\overline{\beta} f(1) - 1).$$

Indeed, by contradiction we assume that $D > \overline{a}(\overline{\beta} f(1) - 1)$. Then, from (4.20) and $\phi(\xi) \to 0$ as $\xi \to -\infty$, there exists $\xi_0 \in \mathbb{R}$ such that $\phi(\xi_0) = D$. Thus, $\phi(\xi_0) = D \geq \phi(\xi)$ for all $\xi \in \mathbb{R}$. Using $\phi(-\infty) = 0$, $\phi \leq 1$, $\psi \leq \overline{\beta} f(1) - 1$, and $f(x) \leq ax$, we obtain that

$$d_1 \widetilde{\mathcal{N}}_1[\phi](\xi_0) - r_1 \phi(\xi_0)[1 - \phi(\xi_0)] + f(1 - \phi(\xi_0))\psi(\xi_0)$$

$$= d_1 \left[\int_{\mathbb{R}} \widetilde{J}_1(x_1) \phi(\xi_0 - x_1) \, dx_1 - \phi(\xi_0) \right] - r_1 \phi(\xi_0)(1 - \phi(\xi_0)) + f(1 - \phi(\xi_0))\psi(\xi_0)$$

$$< d_1 \phi(\xi_0) \int_{\mathbb{R}} \widetilde{J}_1(x_1) \, dx_1 - d_1 \phi(\xi_0) + r_1(1 - \phi(\xi_0))(\overline{a}\psi(\xi_0) - \phi(\xi_0))$$

$$\leq r_1(1 - \phi(\xi_0))[\overline{a}(\overline{\beta} f(1) - 1) - D] \leq 0.$$

According to the first equation of (4.4), we conclude that $\phi'(\xi_0) < 0$ which contradicts $\phi'(\xi_0) = 0$. Hence, (4.21) holds.

From (4.21), (H3) and $\overline{a}\overline{\beta} f(1) < 1$, it follows that the function $V(x,t) = \psi(x \cdot \nu + ct)$ satisfies

$$\begin{cases} \frac{\partial V}{\partial t}(x,t) \geq d_2 \mathcal{N}_2[V(\cdot,t)](x) \\ \qquad\qquad + r_2 V(x,t)[(1 - \overline{a}\overline{\beta} f(1))(\overline{\beta} f(1) - 1) - V(x,t)], & x \in \mathbb{R}^N, \ t > 0, \\ V(x,0) = \psi(x), & x \in \mathbb{R}^N. \end{cases}$$

Then, a similar argument above will deduce that

$$\liminf_{t \to \infty} \psi(\xi) \geq (1 - \overline{a}\overline{\beta} f(1))(\overline{\beta} f(1) - 1) > 0.$$

This completes the proof of the theorem. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following theorem indicates that the nontrivial state at $\xi \to \infty$ is actually a coexisting state, if system (1.1) has a unique coexisting state and the wave tail converges at $\xi \to \infty$.

Theorem 4.7. *Let (U,V) be a traveling wave of (1.1) connecting $(1,0)$ and a nontrivial state. Assume that the limit*

$$(\overline{u}, \overline{v}) := \lim_{\xi \to \infty} (U,V)(\xi)$$

exists. If the kinetic part of system (1.1) has a unique coexistence state (U^*, V^*) satisfying

$$r_1 U^*(1 - U^*) - f(U^*)V^* = 0,$$

$$\beta f(U^*)V^* - r_2 V^*(1 + V^*) = 0,$$

then $(\overline{u}, \overline{v}) = (U^*, V^*)$.

Proof. According to the assumption, we have $(U_t, V_t)(\xi) \to (0,0)$ as $\xi \to \infty$. By (4.1) and priori estimates, we obtain $0 < \overline{u} \le 1$ and $0 < \overline{v} \le \alpha$. To the aim, we only need to show that

$$(\widetilde{J}_1 * U)(\xi) \to \overline{u}, \quad (\widetilde{J}_1 * V)(\xi) \to \overline{v} \quad \text{as } \xi \to \infty,$$

if the kinetic part of system (1.1) has a unique coexistence state (U^*, V^*).

Let $\epsilon \in (0, \overline{u})$. Because $U(+\infty) = \overline{u}$ and $\int_{\mathbb{R}} \widetilde{J}_1(x_1) \, dx_1 = 1$, there exists $K_0 > 0$ large enough such that

$$\int_{K_0}^{\infty} \widetilde{J}_1(x_1) \, dx_1 < \frac{\epsilon}{2} \quad \text{and} \quad \overline{u} - \frac{\epsilon}{2} < U(\xi) < \overline{u} + \frac{\epsilon}{2}, \quad \forall \xi \ge K_0.$$

Denote

$$\int_{\mathbb{R}} \widetilde{J}_1(x_1) U(\xi - x_1) \, dx_1 = \int_{-\infty}^{K_0} \widetilde{J}_1(x_1) U(\xi - x_1) \, dx_1 + \int_{K_0}^{\infty} \widetilde{J}_1(x_1) U(\xi - x_1) \, dx_1$$

$$:= I_3(\xi) + I_4(\xi).$$

Using $\xi - x_1 \ge K_0$ for $x_1 \le K_0$ and $\xi \ge 2K_0$, $\int_{\mathbb{R}} \widetilde{J}_1(x_1) \, dx_1 = 1$ and $U \le 1$, it follows that

$$(4.22) \qquad I_3(\xi) + I_4(\xi) \le \left(\overline{u} + \frac{\epsilon}{2}\right) + \frac{\epsilon}{2} = \overline{u} + \epsilon, \quad \forall \xi \ge 2K_0.$$

On the other hand, since $\overline{u} \le 1$, we have

$$(4.23) \qquad I_3(\xi) + I_4(\xi) \ge I_3(\xi) \ge \left(\overline{u} - \frac{\epsilon}{2}\right)\left(1 - \frac{\epsilon}{2}\right) \ge \overline{u} - \epsilon, \quad \forall \xi \ge 2K_0.$$

By (4.22) and (4.23), we obtain

$$(\widetilde{J}_1 * U)(\xi) \to \overline{u}, \quad \xi \to \infty.$$

Using the same method, we get $(\widetilde{J}_1 * V)(\xi) \to \overline{v}$ as $\xi \to \infty$. This completes the proof. $\quad\square$

4.3. Nonexistence of planar wave solutions for $c < c^*$

Theorem 4.8. *If $c \in (0, c^*)$, then (4.4) does not admit any nontrivial nonnegative solution (ϕ, ψ) satisfying (4.3) and $\lim_{\xi \to -\infty} (\phi(\xi), \psi(\xi)) = (0,0)$.*

Proof. To prove this result, by contradiction we assume that (4.4) admits a nontrivial nonnegative solution (ϕ, ψ) satisfying $\lim_{\xi \to -\infty} (\phi(\xi), \psi(\xi)) = (0,0)$ and (4.3) for some wave speed $c_1 \in (0, c^*)$. Here, we have $\phi > 0$ and $\psi > 0$ on $\mathbb{R}$.

Let $\epsilon \in (0, (\overline{\beta} f(1) - 1)/2)$ be given such that

$$c_2 := \inf_{\lambda > 0} \frac{d_2 \left[\int_{\mathbb{R}} \widetilde{J}_2(x_1) e^{\lambda x_1} \, dx_1 - 1 \right] + r_2(\overline{\beta} f(1) - 1 - \epsilon)}{\lambda} > c_1.$$

Because $\phi(-\infty) = 0$, there exists a $\xi_0 \in \mathbb{R}$ such that $\overline{\beta} f(1)\phi(\xi) < \epsilon$ for all $\xi \leq \xi_0$. Due to $\psi > 0$ as $\xi > \xi_0$ and (4.3), there exists $\delta > 0$ such that $\inf_{\xi > \xi_0} \psi(\xi) \geq \delta$.

Define $M_2 > 1$ by $\delta(M_2 - 1) = \overline{\beta} f(1)$. Then, since $\phi \leq 1$, we get

$$\overline{\beta} f(1)\phi(\xi) \leq (M_2 - 1)\psi(\xi) \quad \text{for all } \xi \geq \xi_0.$$

Using (H3), we deduce that

$$c_1 \psi'(\xi) \geq d_2 \widetilde{\mathcal{N}}_2[\psi](\xi) + r_2\psi(\xi)[\overline{\beta} f(1) - 1 - \epsilon - M_2\psi(\xi)], \quad \xi \in \mathbb{R},$$

and the function $V(x,t) := \psi(x \cdot \nu + c_1 t)$ satisfies

$$\begin{cases} \frac{\partial V}{\partial t}(x,t) \geq d_2\mathcal{N}_2[V(\cdot,t)](x) + r_2 V(x,t)[\overline{\beta} f(1) - 1 - \epsilon - M_2 V(x,t)], & x \in \mathbb{R}^N, \ t > 0, \\ V(x,0) = \psi(x) > 0, & x \in \mathbb{R}^N. \end{cases}$$

Denote $c := (c_1 + c_2)/2 < c_2$. Applying Propositions 2.1 and 2.3(ii), we can infer that

(4.24)
$$\liminf_{t \to \infty} V_{\|x\| < ct}(x,t) \geq \frac{\overline{\beta} f(1) - 1 - 2\epsilon}{M_2} > 0.$$

Letting $x_1 = -\frac{c_1 + c_2}{2} t e_1$, where $e_1 = (1, 0, \ldots, 0)$, we have $\|x_1\| < ct$. Clearly,

$$-(c_2 + c_1)t/2 + c_1 t = (c_1 - c_2)t/2 \to -\infty \quad \text{as } t \to \infty.$$

Thus, it follows from (4.24) that

$$\liminf_{\xi \to -\infty} \psi(\xi) = \liminf_{t \to \infty} V\left(-\frac{c_1 + c_2}{2} t, t\right) \geq \frac{\overline{\beta} f(1) - 1 - 2\epsilon}{M_2} > 0,$$

which contradicts the limit condition $\psi(-\infty) = 0$. This completes the proof. $\qquad \square$

4.4. Minimal wave speed

In this subsection, we conclude that the spreading speed of the predatory is the minimal wave speed c^*.

Theorem 4.9. *In addition to (1.3), assume that $a\beta f(1) < r_1 r_2$ and $d_2 < \beta f(1) - r_2$, and that J_2 has a compact support. Then, (1.1) admits planar wave solution connecting $(1, 0)$ to a nontrivial state with speed $c \in \mathbb{R}$ if and only if $c \geq c^*$.*

Proof. According to the results in Sections 4.2 and 4.3, in order to prove the theorem, we only need to prove that, under the parameter condition $d_2 < \beta f(1) - r_2$, the nonnegative planar wave solution (ϕ, ψ) of system (4.4) combined with $(\phi, \psi)(-\infty) = (0, 0)$ and (4.3) satisfies $c > 0$. Indeed, by contradiction, we assume that (4.4) admits a nonnegative solution (ϕ, ψ) with speed $c \leq 0$ satisfying $(\phi, \psi)(-\infty) = (0, 0)$ and (4.3).

Due to $(\phi, \psi)(-\infty) = (0, 0)$ and (H3), there exists a sufficiently large constant M_3 such that for all $\xi < -M_3$,

(4.25)
$$r_2[\overline{\beta}f(1 - \phi(\xi)) - 1 - \psi(\xi)] \geq r_2[\overline{\beta}f(1) - \overline{\beta}f(1)\phi(\xi) - 1 - \psi(\xi)]$$
$$> L_1 := \frac{r_2(\overline{\beta}f(1) - 1)}{2} + \frac{d_2}{2}.$$

Since $\widetilde{J}_2 \geq 0$ and $\psi > 0$, it follows from (4.4) and (4.25) that for all $\xi < -M_3$,

$$c\psi'(\xi) = d_2 \int_{\mathbb{R}} \widetilde{J}_2(\xi - x_1)\psi(x_1)\,dx_1 - d_2\psi(\xi) + r_2\psi(\xi)[\overline{\beta}f(1 - \phi(\xi)) - 1 - \psi(\xi)]$$
$$\geq r_2\psi(\xi)[\overline{\beta}f(1) - \overline{\beta}f(1)\phi(\xi) - 1 - \psi(\xi)] - d_2\psi(\xi)$$
$$\geq L_1\psi(\xi) - d_2\psi(\xi)$$
$$= (L_1 - d_2)\psi(\xi).$$

Moreover, according to the condition $d_2 < \beta f(1) - r_2 = r_2(\overline{\beta}f(1) - 1)$, we have $L_2 := L_1 - d_2 > 0$. Integrating from $-\infty$ to $y \leq -M_3$ for the above inequality, we have

$$c\psi(y) \geq L_2 \int_{-\infty}^{y} \psi(\xi)\,d\xi, \quad \forall\, y \leq -M_3.$$

Thus, we have $c > 0$ which contradicts $c \leq 0$. This completes the proof. $\square$

5. Numerical simulations

In this section, we illustrate the main results by using numerical simulations.

For simplicity, we consider $N = 1$, $f(U) = \frac{U}{1+U}$, that is, the system (1.1) becomes

(5.1)
$$\begin{cases} \frac{\partial U}{\partial t}(x, t) = d_1 \mathcal{N}_1[U(\,\cdot\,, t)](x) + r_1 U(x, t)(1 - U(x, t)) \\ \qquad\quad - \frac{U(x,t)}{1+U(x,t)}V(x, t), \\ \frac{\partial V}{\partial t}(x, t) = d_2 \mathcal{N}_2[V(\,\cdot\,, t)](x) + \beta \frac{U(x,t)}{1+U(x,t)}V(x, t) \\ \qquad\quad - r_2 V(x, t)(1 + V(x, t)), \end{cases} \quad x \in \mathbb{R},\ t > 0.$$

We take the kernel function $J_i(x) = J_\varrho(x) = \frac{1}{\sqrt{2\pi\varrho^2}}e^{\frac{-x^2}{2\varrho^2}}$ $(i = 1, 2)$. In addition, the parameters of system (5.1) are taken as: $d_1 = d_2 = 1$, $r_1 = 4$, $r_2 = 1$, $\varrho = 1$, $\beta = 5$. It is easy to know that the system (5.1) has three constant equilibrium states: $E_0 = (0, 0)$, $E_1 = (1, 0)$ and $E^* = (U^*, V^*) = (0.8271, 1.2634)$.

First, we use numerical simulations to illustrate the spreading speed of the predator. We choose $\Omega_1 = [-150, 150]$ and initial conditions

(5.2)
$$U(x, 0) = 1, \quad v_0(x) = \begin{cases} \frac{1}{5}, & x = 0, \\ \frac{1}{5}\left(\frac{1}{5} - |x|\right), & 0 < |x| \leq 5, \\ 0, & |x| > 5. \end{cases}$$

With the help of MATLAB, we obtain the numerical solution (U, V) of system (5.1) under the initial conditions (5.2) (see Figure 5.1). Figure 5.1 shows that there are a small number of predators V near the origin. These predators spread in space over time and feed on the aborigine prey U (the density of U is 1), after a period of time, the density of prey decreases and the density of the predator increases. If let $x_\pm^a(t)$ denote the spatial position when the species V diffuses to the density a (when spreading to the left to the density a, it is represented by $x_-^a(t)$, and to the right is represented by $x_+^a(t)$), as shown in Figure 5.2. The spreading speed can be defined as $c^* = \pm \lim_{t \to \infty} \frac{dx_\pm^a(t)}{dt}$, $\forall a \in (0, 1)$. From Figures 5.2 and 5.3, we get the spreading speed of the predator is about $c^* = 2.118$.
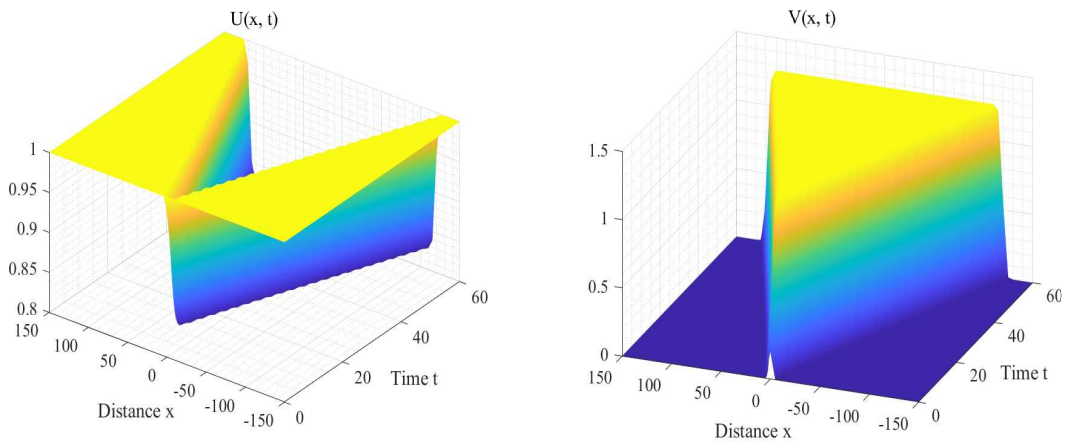
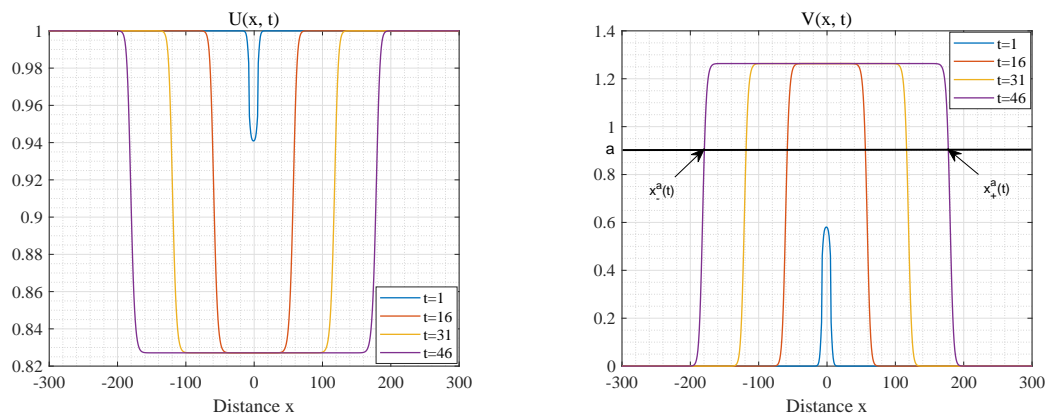Figure 5.1: Numerical solution of system (5.1) under initial conditions (5.2).

Figure 5.2: Figures of the solution in Figure 5.1 at time $t = 1, 16, 31, 46$.

Now, we use numerical simulations to study the planar wave solutions of system (5.1).

We choose $\Omega_2 = [-400, 400]$ and the initial datum

(5.3)
$$
\begin{cases}
U(x,0) = (1 - U^*)\big(1 - \frac{1}{4}(1 + \tanh(x))^2\big) + U^*, \\
v_0(x) = V^*\big(1 - \frac{1}{4}(1 - \tanh(x))^2\big),
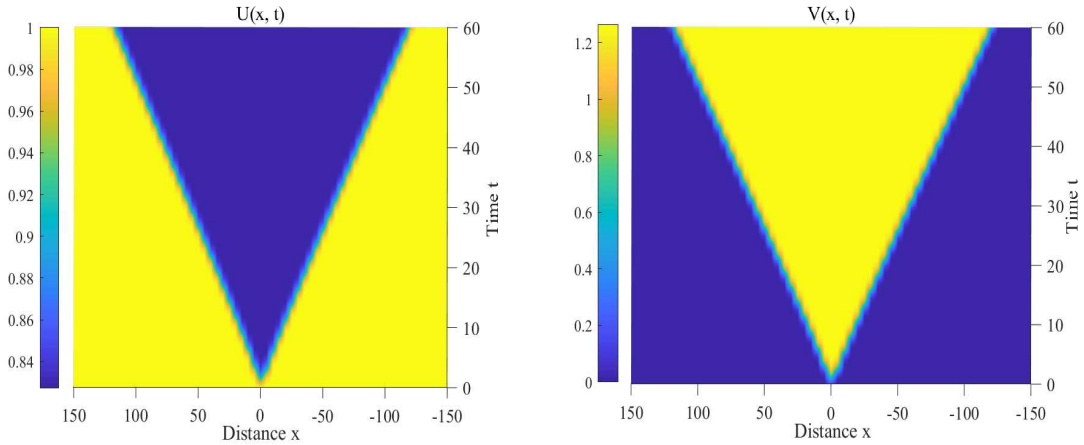\end{cases}
\qquad x \in [-400, 400].
$$

Figure 5.3: The projections of $U(x,t)$ and $V(x,t)$ on the xot plane in Figure 5.1.

From Theorems 4.4, 4.5 and 4.6, system (5.1) admits a planar wave connecting the predator-free state $(1,0)$ and a nontrivial state with speed $c \geq c^* = 2.118$. According to Theorem 4.9, we know that c^* is also the minimal wave speed of the invasion planar waves. Through numerical simulations (see Figure 5.4), we can get the same result as Theorem 4.7, namely, the solution of system (5.1) eventually converge to the positive equilibrium (U^*, V^*), which implies that the species U and V will coexist.
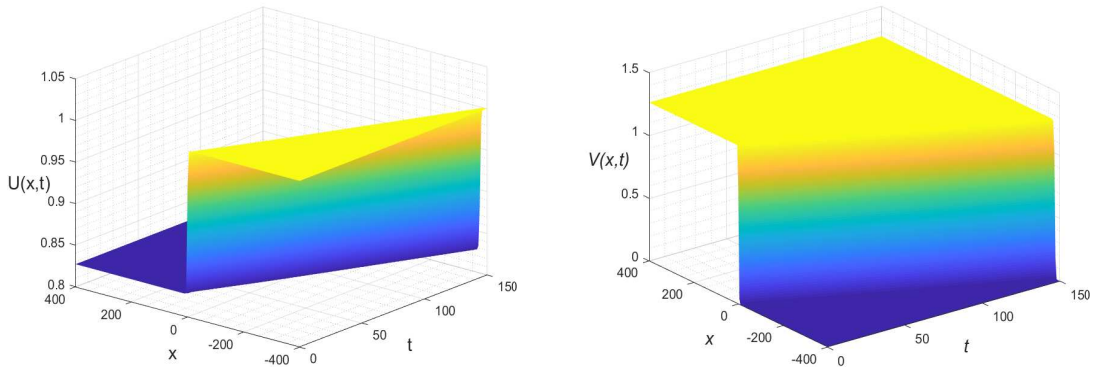

Figure 5.4: The solutions of system (5.1) with initial (5.3).

Here, we give some explanations of the ecological phenomena related to these results. In the actual biological environment, at the beginning, there was only the aborigine species

U with a density at its carrying capacity. After some time, the exotic species V invades the system (5.1) by feeding on species U. Then based on our results, the two species will eventually coexist.

Acknowledgments

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