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Spreading speeds for the predator-prey system with nonlocal dispersal

Min Zhao [a,*], Rong Yuan [a], Zhaohai Ma [b], Xiao Zhao [a]

[a] *Laboratory of Mathematics and Complex Systems (Ministry of Education), School of Mathematical Sciences, Beijing Normal University, Beijing 100875, People's Republic of China*

[b] *School of Science, China University of Geosciences, Beijing 100083, People's Republic of China*

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Abstract

In this paper, we mainly study the propagation properties of solutions of the predator-prey system with nonlocal dispersal. More specifically, we explored the spreading speeds of the predator and the prey in two different situations, namely, the predator spreads faster than the prey and the predator spreads slower than the prey. The main difficulty lies in the fact that the comparison principle cannot be used for the predator-prey system. We use the comparison principle of the scalar equation and the method of upper and lower solutions to prove the results. In addition, we establish the comparison principle of nonlocal dispersal equations in space and time dependent environment. We conclude that the predator and the prey will eventually coexist by constructing a suitable Lyapunov functional. Finally, we use numerical simulations to illustrate the results.

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Keywords: Predator-prey system; Spreading speed; Nonlocal dispersal; Comparison principle; Lyapunov functional

* Corresponding author.
E-mail addresses: minzhao@mail.bnu.edu.cn (M. Zhao), ryuan@bnu.edu.cn (R. Yuan), zhaohaima@cugb.edu.cn (Z. Ma), xzhao@mail.bnu.edu.cn (X. Zhao).

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1. Introduction

The population dynamics of each species will be affected by species interactions [41]. There are three main types of interactions when considering a system of two or more species, and especially when focusing on a system of two species. The first type is called a predator-prey system if the growth rate of one population (called the predator) is an increasing function of the other species (called the prey) abundance, while the growth rate of the prey population is a decreasing function of predator species abundance; the second one is called competition if the growth rate of each population is an decreasing function of the other species abundance; the third is called cooperation if the growth rate of each population is an increasing function of the other species abundance [7,41]. The predator-prey relationship among different species is widespread in a variety of ecosystems, such as marine ecosystems, terrestrial ecosystems, island ecosystems, and so on [31]. It is crucial to study the dynamics of the interactions between the predator and the prey in theoretical ecology [22].

In nature, predation exists in various species, such as mammals, birds, fish, insects and bacteria, etc. [47]. There are many examples of the relationship between the predator and the prey, especially the biological invasion of the predator. A classic example is that in Central Europe from 1909 to 1927, muskrats reproduced by capturing cattails, freshwater mussels, frogs, freshwater crayfish and small tortoises [47]. Another example is the invasion of the American continent by European starlings (Sturnus vulgaris) native to Europe, Asia and North Africa [43,47]. It is an omnivorous bird that feeds mainly on seeds, insects, invertebrates, plants and fruits. When they were artificially introduced into the American continent, they began to multiply, and are now distributed all over the world [43,47]. The third one is the jellyfish Mnemiopsis leidyi (commonly called Mnemiopsis) that invaded the Black Sea in the early 1980s. It is a pelagic fish native to the coastal waters of the United States along the Atlantic Ocean and feeds on eggs and larvae. Lack of predators for certain fishes and favorable environmental conditions (rich food, natural environment similar to the local areas) encouraged Mnemiopsis establishment in the Black Sea [25,31,46].

The interactions between the predator and the prey have become one of the main subjects of biological mathematics [47,48]. Pioneer work on this aspect began with Lotka [39] and Volterra [52], who proposed the Lotka-Volterra predator-prey model in 1921 and 1923, respectively. Bazykin [6] proposed a predator-prey model that included mortality due to predator density constraints in 1976. This type of mortality can regulate or limit the infinite growth of the predator [18]. For most complex predation processes in nature, affected by environmental factors, the predator and the prey have intraspecific competition, which means that the predator and the prey have density constraints [5,27]. In this paper, we mainly study the density-constrained predator-prey system with logistic growth, which is more in line with biological significance.

Holling [20] proposed three Holling-type functional responses in 1965, which are very meaningful and practical. Different species can be described by different functional response functions. Holling type II (Michaelis-Menten) functional response is suitable for some carnivorous fish or invertebrates. In order to simulate the predation behavior more realistically, we study the propagation dynamics of the predator-prey system with Holling type II.

In a real environment, the predator and its prey are distributed in space. As the predator moves to capture the prey and the prey escapes from the predator, temporal and spatial changes in the population will occur [42]. In general, we use dispersal to describe the movement of organisms from one location to another, and this dispersal phenomenon exists in almost all biological and ecology systems [4,24]. There are two important forms of dispersal in population dynamics, namely random dispersal, and nonlocal dispersal [4].

Random dispersal is governed by random walk, which is essentially a local behavior, describing that creatures can only move to their surrounding neighborhoods [4,24]. In this way, we can describe the dynamics of random dispersal through the reaction-diffusion model. In general, the differential operators (such as Δu) are used to model random dispersal. Many scholars have conducted a lot of research on biological and ecological models with random dispersal [8,10,14,47,49]. Ducrot et al. [14] mainly explored the propagation properties of a class of predator-prey systems with random dispersal.

However, some organisms in the ecosystem can travel for some distance, and their movement and interactions may occur between non-adjacent spatial locations [4,24,28,29]. This kind of dispersal is called nonlocal dispersal and is usually modeled by an appropriate integral operator, such as $\int_{\mathbb{R}} J(x-y)[u(y)-u(x)]\mathrm{d}y$. It is very important to study biological and population models of nonlocal dispersal. There are many studies on nonlocal dispersal population dynamics models to explain the spatial variation of interactions between different populations in biology, see [12,13,33,55]. The purpose of this paper is to study the predator and the prey co-invasions of the following Lotka-Volterra predator-prey system with nonlocal dispersal,

$$\begin{cases} \frac{\partial U}{\partial t}(x,t) = d_1[(J_1 * U)(x,t) - U(x,t)] + r_1 U(x,t)(1-U(x,t)) - \frac{bU(x,t)V(x,t)}{1+\alpha U(x,t)}, \\ \frac{\partial V}{\partial t}(x,t) = d_2[(J_2 * V)(x,t) - V(x,t)] + \frac{\beta b U(x,t)V(x,t)}{1+\alpha U(x,t)} - aV(x,t) - aV^2(x,t), \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}, t > 0$ and r_1, b, β, a, α are all positive constants. Here $U(x,t)$ and $V(x,t)$ are the population densities of prey and predator species at spatial position $x \in \mathbb{R}$ and time $t > 0$, respectively. The dynamics of the prey population follow a logistic growth with a normalized (to one) carrying capacity and r_1 denotes its intrinsic growth rate. The function $\frac{U(x,t)V(x,t)}{1+\alpha U(x,t)}$ describes predation and is called the predator functional response to prey, where the parameter α measures the "satiation" effect of the predator population. The constant b denotes the predation rate and β denotes the biomass conversion rate coefficient. Thus βb denotes the biomass conversion rate. The parameter a denotes the death rate of the predator. $d_1 > 0$ and $d_2 > 0$ are the diffusion coefficients for prey and predator species, respectively. The term $J_i * w - w$ describes the spatial dispersal process and

$$(J_i * w)(x,t) - w(x,t) = \int_{\mathbb{R}} J_i(x-y)w(y,t)\mathrm{d}y - w(x,t), \quad i = 1, 2,$$

where the symbol $*$ denotes the convolution product for the spatial variable. Here we assume that the kernel function $J_i : \mathbb{R} \to \mathbb{R} (i = 1, 2)$ is continuous and satisfies the following properties:

(J1) $J_i(x) = J_i(-x) \geq 0$ for any $x \in \mathbb{R}$ and $\int_{\mathbb{R}} J_i(x)\mathrm{d}x = 1, i = 1, 2$;

(J2) $\int_{\mathbb{R}} J_i(x)\mathrm{e}^{\lambda x}\mathrm{d}x < \infty$ for any $\lambda \geq 0$, i=1, 2;

(J3) $J_i \in C^1(\mathbb{R})$ and J_i is compactly supported, i=1, 2;

(J4) there exists $\varrho > 0$ such that $J_i(x) \geq \varrho$ for a.a. $x \in (-\varrho, \varrho)$, $i = 1, 2$.

System (1.1) is supplemented by the initial conditions:

$$U(x,0)=u_0(x), \quad V(x,0)=v_0(x), \quad x\in\mathbb{R}, \tag{1.2}$$

where $u_0(x)$, $v_0(x)$ are bounded nonnegative functions with the nonempty compact support. In addition, we assume that $u_0(x)$ and $v_0(x)$ are differentiable with respect to x, which ensures that the solutions of system (1.1) are differentiable with respect to x. Since $u_0(x)$ and $v_0(x)$ have nonempty compact support, the propagation of both species occurs in an initially empty environment. This is similar to the “hair-trigger effect” of the scalar monostable equation [2]. In this paper, both predator and prey can be regarded as alien species invading into the initially empty environment. This phenomenon may occur in biological control, especially in the prevention and control of the invasion of alien species [31]. Predation can affect the spread of alien species, thus predation can be used as a biological control agent [31]. For example, Fagan and Bishop [15] studied that herbivores slowed a plant reinvasion at Mount St. Helens after the eruption of Mount St. Helens in 1980.

Throughout this work, we give the following assumption about the parameters:

(H1) $\quad r_2 := \frac{\beta b}{1+\alpha} - a > 0,$

which ensures that the amount of prey is sufficient to maintain the positive density of predator. From system (1.1), we can see that the predator cannot survive without the prey.

At present, there are many studies on spreading speeds and the existence of traveling wave solutions of the reaction-diffusion equations [1,2,11,14,16,17,26,35,40,53–55]. In particular, Kolmogorov, Petrovsky, and Piskunov [26], Fisher [17] studied the front propagation of Fisher-KPP type equations. Aronson and Weinberger [1,2] later proposed the concept of the spreading speed and studied the continuous and discrete time models. We refer to Li et al. [32], Lewis et al. [30], and Hu et al. [21] for the spreading speeds of cooperative systems. We refer to [36,44,45] for studies of the predator invasion with usual linear dispersal, to [14] for a study of the predator and the prey co-invasions with random dispersal. However, there are few research on the predator and the prey co-invasions with nonlocal dispersal. We refer to Zhang and Zhao [55] for the spreading speeds of a two-species strong competition system with nonlocal dispersal using the comparison principle of the system and the method of upper and lower solutions.

In this paper, we mainly consider the long-term behavior of the solutions of the predator-prey system with nonlocal dispersal, namely spreading speeds of the solutions of system (1.1) with initial conditions (1.2). The comparison principle of the nonlocal reaction-diffusion equations plays an important role in studying the argument. For nonlocal reaction-diffusion equations, Jin and Zhao [23] established the comparison principle in time periodic environment, Kao, Lou and Shen [24] mainly studied the comparison principle with specific kernel function in space periodic environment, Bates et al. [3] and Li et al. [34] respectively gave the comparison principle in space and time dependent environment. Based on the needs of research and the assumptions of the kernel function, we establish the general comparison principle of nonlocal reaction-diffusion equations in space and time dependent environment.

In addition, for the predator-prey system (1.1), the increase in the amount of predators will reduce the population of preys, while the increase in the population of preys will increase the amount of predators. Due to this asymmetry, the comparison principle and the theory related to monotone semiflows [37,38] cannot be applied in system (1.1). Therefore, this poses certain difficulties in studying the spreading properties of predator-prey systems. Inspired by the work of

Ducrot et al. [14] and Zhang et al. [55], we use the global boundedness property and the method of upper and lower solutions to study the asymptotic speed of propagation of system (1.1). In addition, we study the asymptotic behavior of the solution of system (1.1) by constructing a suitable Lyapunov functional. We conclude that under certain conditions, the predator and the prey respectively tend to a nontrivial equilibrium solution, that is, the two species coexist. This phenomenon is very common in nature, but it is difficult to verify due to the lack of comparison principle for the system and the nonlocal dispersal. Finally, we use MATLAB to simulate the propagation of the predator and the prey in two cases, namely the slow predator and the fast predator.

This paper is organized as follows. In the next section, we establish some preliminary results. In Section 3, we mainly consider the asymptotic speed of propagation of the predator and the prey in the case that the predator spreads slower than the prey. We get the results by constructing appropriate upper and lower solutions and using the comparison principle of the scalar equation. In Section 4, we consider the propagation behavior when the predator spreads faster than the prey. We study the asymptotic behavior of solutions by constructing the Lyapunov functional in Section 5. In Section 6, we use numerical simulations to illustrate the main results of this paper.

2. Preliminaries

In this section, we mainly introduce some preliminaries and main results. In subsection 2.1, we give the comparison principle of the nonlocal reaction diffusion equation which is essential to prove the spreading speeds of the predator and the prey (i.e. Theorems 3.1 and 4.1). Then, we give some results on spreading speeds and the existence of traveling wave solutions of the nonlocal reaction-diffusion equations in subsection 2.2.

Firstly, we define some space aspects. Let

$$X = \{w(x) |\ w(x) : \mathbb{R} \to \mathbb{R} \text{ is bounded and uniformly continuous}\},$$

with the norm

$$\|w\|_X = \sup_{x\in\mathbb{R}} |w(x)|.$$

Then $(X, \|\cdot\|_X)$ is a Banach space. The positive cone X^+ is defined by

$$X^+ = \{w \in X : w(x) \geq 0,\ \forall x \in \mathbb{R}\}.$$

Furthermore, for any constant $d > 0$, let

$$X_d = \{w \in X : 0 \leq w(x) \leq d,\ \forall x \in \mathbb{R}\}.$$

Set the order of the space $X^2 = X \times X$ as follows:

$$\underline{w} \leq \overline{w} \Leftrightarrow \underline{w}_i(x) \leq \overline{w}_i(x), \quad x \in \mathbb{R}, i = 1, 2,$$

for any $\underline{w} = (\underline{w}_1(x), \underline{w}_2(x))$ and $\overline{w} = (\overline{w}_1(x), \overline{w}_2(x)) \in X^2$.

We define the set $H \subset X^2$ by

$$H = \{(w_1, w_2) \in X^2 : 0 \le w_1 \le 1 \text{ and } 0 \le w_2 \le \hat{r}_2\},$$

where $\hat{r}_2 := \frac{r_2}{a}$. Our initial datum will always be chosen in the set of H. Here, we point out that the set H is positively invariant under the semiflow $\{S(t)\}_{t\ge 0}$ generated by system (1.1). In particular, this means that system (1.1) with initial condition (1.2) admits a unique globally defined solution $(U(x,t), V(x,t))$ with

$$(U, V)(x, \cdot) \in C^1([0,\infty), X^2), \quad \forall x \in \mathbb{R} \quad \text{and} \quad (U, V)(\cdot, t) \in H, \quad \forall t \ge 0.$$

2.1. *Comparison principle*

The comparison principle of nonlocal diffusion equations is crucial in the study of the spreading speeds of system (1.1). Jin and Zhao [23] studied a scalar periodic logistic equation with nonlocal diffusion and established the comparison principle. Kao, Lou and Shen [24] established the comparison principle of the nonlocal diffusion equation in spatial periodic environment. Bates et al. [3] and Li et al. [34] studied the comparison principle of nonlocal diffusion equations in space or time dependent environment, respectively. Different from the above three works, we give the comparison principle of nonlocal diffusion system in space-time dependent media when the kernel function J satisfies (J1) and (J4), see Theorem 2.1. In particular, the assumptions of the kernel function J in Theorem 2.1 are different from Bates and Chen's work [3], thus the corresponding proof of Theorem 2.1 is not the same.

Theorem 2.1. *(Comparison principle) Assume that J satisfies* (J1) *and* (J4), $u \in C^1([\tau, t_0], X)$. *Suppose that $u(x,t)$ satisfies*

$$\begin{cases} u_t - c_0 u_x - K_0(x)u - d(J * u - u) \ge 0, & (x,t) \in \mathbb{R} \times (\tau, t_0], \\ u(x,\tau) \ge 0, & x \in \mathbb{R}, \end{cases}$$

where $K_0(\cdot) \in X$, $c_0 \ne 0$, $\tau < t_0 \in \mathbb{R}$; $d > 0$ and c_0 are constants. Then $u(x,t) \ge 0$ on $\mathbb{R} \times [\tau, t_0]$. Moreover, if $u(x,\tau) \not\equiv 0$ on $x \in \mathbb{R}$, then $u(x,t) > 0$ on $\mathbb{R} \times (\tau, t_0]$.

Proof. We may assume $\tau = 0$. Let $\bar{u}(x,t) = u(x - c_0 t, t)$. Then $\bar{u}(x,t)$ satisfies

$$\begin{cases} \bar{u}_t - \bar{K}_0(x,t)\bar{u} - d(J * \bar{u} - \bar{u}) \ge 0, & (x,t) \in \mathbb{R} \times (0, t_0], \\ u(x,0) \ge 0, & x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where $\bar{K}_0(x,t) := K_0(x - c_0 t)$ and $\bar{K}_0(\cdot, t) \in X$ for any $t \in [0, t_0]$. As u is continuous, the function $f(t) := \inf_{x\in\mathbb{R}} \bar{u}(x,t)$ is continuous in $[0, t_0]$. Set $g(t) = \mathrm{e}^{-2Kt} f(t)$ where $K := 2d + \|K_0\|_X$. If the first conclusion of Theorem 2.1 is not true and due to $g(0) \ge 0$, then there exist constants $\epsilon > 0$, $T \in (0, t_0]$ such that $g(T) = \inf_{t\in[0,t_0]} g(t) = -\epsilon$, $g(t) > -\epsilon$ for $0 \le t < T$, and there exists $(x_*, t_*) \in \mathbb{R} \times (0, T)$ such that $\bar{u}(x_*, t_*) < -\frac{31}{32}\epsilon \mathrm{e}^{2Kt_*}$. Therefore,

$$\bar{u}(x,t) > -\epsilon e^{2Kt} \text{ in } \mathbb{R} \times [0, T), \quad \bar{u}(x_*, t_*) < -\frac{31}{32}\epsilon \mathrm{e}^{2Kt_*}.$$

Let $z(x)$ be a smooth function in $\mathbb{R}$ satisfying $\min_{x\in\mathbb{R}} z(x) = z(x_*) = 1$, $\sup_{x\in\mathbb{R}} z(x) = z(\pm\infty) = 3$ and $\left|z'(x)\right| \le 1$. Define

$$w_\sigma(x,t) = -\epsilon\left(\frac{3}{4} + \sigma z(x)\right)\mathrm{e}^{2Kt} \text{ for } \sigma \in [0,1].$$

Clearly, w_σ is bounded and continuous in $[0,1]\times\mathbb{R}\times[0,t_0]$. Notice that, when $\sigma \le 1/8$,

$$\inf_{(x,t)\in\mathbb{R}\times[0,T]}(\bar{u} - w_\sigma)(x,t) \le \bar{u}(x_*,t_*) - w_\sigma(x_*,t_*) < -\frac{31}{32}\epsilon\mathrm{e}^{2Kt_*} + \epsilon\left(\frac{3}{4}+\frac{1}{8}\right)\mathrm{e}^{2Kt_*}$$
$$= -\frac{3}{32}\epsilon\mathrm{e}^{2Kt_*};$$

when $\sigma > 1/4$, for all $(x,t)\in\mathbb{R}\times[0,T]$,

$$\bar{u}(x,t) - w_\sigma(x,t) \ge -\epsilon\mathrm{e}^{2Kt} + \epsilon\left(\frac{3}{4}+\sigma z(x)\right)\mathrm{e}^{2Kt} \ge \epsilon\left(\frac{3}{4}+\sigma-1\right)\mathrm{e}^{2Kt} \ge \epsilon\left(\sigma - \frac{1}{4}\right).$$

Thus, there is a minimum $\sigma^* \in \left(\frac{1}{8},\frac{1}{4}\right]$ such that $w_{\sigma^*}(x,t) \le \bar{u}(x,t)$ for $(x,t)\in\mathbb{R}\times[0,T]$. We claim that there exist $(x_n,t_n)\in\mathbb{R}\times[0,T]$ and $\left(\bar{x}_0,\bar{t}_0\right)$ such that

$$\lim_{n\to\infty}(x_n,t_n) = (\bar{x}_0,\bar{t}_0);$$
$$\bar{u}_t(x_n,t_n) - \bar{K}_0(x_n,t_n)\bar{u}(x_n,t_n) - d(J*\bar{u}-\bar{u})(x_n,t_n) \ge 0;$$
$$\lim_{n\to\infty}[\bar{u}(x_n,t_n) - w_{\sigma^*}(x_n,t_n)] = 0 = \inf_{(x,t)\in\mathbb{R}\times[0,T]}[\bar{u}(x,t) - w_{\sigma^*}(x,t)];$$

and

$$\lim_{n\to\infty}(\bar{u} - w_{\sigma^*})_t(x_n,t_n) \le 0.$$

Indeed, let $v(x,t) = \bar{u}(x,t) - w_{\sigma^*}(x,t)$ and $\rho(t) = \inf_{x\in\mathbb{R}} v(x,t)$. Then $\rho(0) > 0$. Let $\bar{t}_0 = \max\{t\in(0,T] : \rho(\tau) > 0 \text{ for all } 0 \le \tau < t\}$. Note that $w_{\sigma^*}(\pm\infty,t) \le -\frac{9}{8}\epsilon e^{2Kt} < \bar{u}(\pm\infty,t)$ for $t\in[0,T]$ and $\bar{u}(x,0) \ge 0 > -3\epsilon/4 > w_{\sigma^*}(x,0)$ for $x\in\mathbb{R}$. For each $t < \bar{t}_0$, since $\rho\left(\bar{t}_0\right) = 0 < \rho(t)$, there is a uniformly bounded set $A\left(\bar{t}_0,t\right)\subset\mathbb{R}$ such that $v\left(x,\bar{t}_0\right) \le \rho(t)$ and $\bar{u}$ satisfies inequality (2.1) for all $x\in A\left(\bar{t}_0,t\right)$. Therefore, for $\bar{t}_n = \bar{t}_0 - \frac{1}{n}$, we have $0 \le v(x,\bar{t}_n) - v(x,\bar{t}_0) = \int_0^1 v_t(x,\bar{t}_0 + s(\bar{t}_n - \bar{t}_0))\mathrm{d}s(\bar{t}_n - \bar{t}_0)$ for $x\in A(\bar{t}_0,\bar{t}_n)$. Therefore, there exist $t_n\in(\bar{t}_n,\bar{t}_0)$ and a bounded sequence $x_n\in A(\bar{t}_0,\bar{t}_n)$ such that $v_t(x_n,t_n) \le 0$. After taking a subsequence of x_n, we assume that the limit of x_n exists and $\bar{x}_0 := \lim_{n\to\infty} x_n$. Then $|v(x_n,t_n)| \le \left|v\left(x_n,\bar{t}_0\right)\right| + \left|v(x_n,t_n) - v\left(x_n,\bar{t}_0\right)\right| \le \left|v\left(x_n,\bar{t}_0\right)\right| + \|v_t\|_X\left(\bar{t}_0 - t_n\right) \to 0$ where we have used the fact that $\left|v\left(x_n,\bar{t}_0\right)\right| \le \rho\left(\bar{t}_n\right) \to 0$. This proves the claim.

Therefore,

$$
\begin{aligned}
0 &\geq \lim_{n\to\infty} (\bar{u} - w_{\sigma^*})_t \,(x_n, t_n) \\
&\geq d(J * \bar{u})\left(\bar{x}_0, \bar{t}_0\right) - (d - \bar{K}_0(x,t))\bar{u}(\bar{x}_0, \bar{t}_0) + 2K\epsilon \mathrm{e}^{2K\bar{t}_0}\left(\sigma^* z\left(\bar{x}_0\right) + \frac{3}{4}\right) \\
&\geq -d\epsilon \mathrm{e}^{2K\bar{t}_0} - (d + \|\bar{K}_0\|_X)|\bar{u}(\bar{x}_0, \bar{t}_0)| + \frac{7}{4}K\epsilon \mathrm{e}^{2K\bar{t}_0} \\
&\geq -d\epsilon \mathrm{e}^{2K\bar{t}_0} - \frac{3}{2}\epsilon(d + \|\bar{K}_0\|_X)\mathrm{e}^{2K\bar{t}_0} + \frac{7}{4}K\epsilon \mathrm{e}^{2K\bar{t}_0} \\
&= \epsilon e^{2K\bar{t}_0}\left[\frac{7}{4}K - d - \frac{3}{2}(d + \|\bar{K}_0\|_X)\right] > 0
\end{aligned}
$$

which is a contradiction. Therefore $u(x,t) \geq 0$ on $(x,t) \in \mathbb{R} \times [\tau, t_0]$.

Now we give the proof of the last part of the theorem. Let $\bar{v}(x,t) = e^{Kt}\bar{u}(x,t)$ (K is given above, i.e. $K = 2d + \left\|\bar{K}_0\right\|_X$). Since $\bar{u}(x,t)$ satisfies (2.1), then we have

$$
\bar{v}_t(x,t) - (K + \bar{K}_0(x,t))\bar{v}(x,t) - d(J * \bar{v} - \bar{v})(x,t) \geq 0, \quad (x,t) \in \mathbb{R} \times (0, t_0]. \tag{2.2}
$$

If $u(x,0) \not\equiv 0$ in $\mathbb{R}$, then we have $v(x,0) \geq, \not\equiv 0$, for $x \in \mathbb{R}$.

To complete the proof, it suffices to show that $u(x,t) > 0$ for $(x,t) \in \mathbb{R} \times (0, t_0]$. Suppose that there exist some $x^* \in \mathbb{R}, t^* > 0$ such that

$$
\bar{v}\left(x^*, t^*\right) = 0. \tag{2.3}
$$

Here $t^* \neq 0$, otherwise the initial function $u(x,0)$ may equal to 0. Thus, $t^* = 0$ is not considered. Since $\bar{v}(x,t) \geq 0$ on $\mathbb{R} \times (0, t_0]$, then we have

$$
\bar{v}\left(x^*, t^*\right) \leq \bar{v}(x,t) \text{ for } (x,t) \in \mathbb{R} \times (0, t_0]. \tag{2.4}
$$

It follows from (2.4) that

$$
\left.\frac{\partial \bar{v}}{\partial t}\left(x^*, t\right)\right|_{t = t^{*-}} \leq 0. \tag{2.5}
$$

By (2.2), (2.3), (J1) and recall that $K = 2d + \left\|\bar{K}_0\right\|_X$, we obtain that

$$
\begin{aligned}
\bar{v}_t(x^*, t^*) &\geq (K + \bar{K}_0(x^*, t^*) - d)\bar{v}(x^*, t^*) + d(J * \bar{v})(x^*, t^*) \\
&\geq 0.
\end{aligned} \tag{2.6}
$$

Then (2.5) and (2.6) imply that

$$
\bar{v}_t\left(x^*, t^*\right) = 0. \tag{2.7}
$$

It follows from (2.2) and (2.7) that

$$d(J*\bar{v}-\bar{v})\left(x^*,t^*\right)\le 0.$$

On the hand, using (2.4) and (J1), we have that

$$\begin{aligned} d(J*\bar{v}-\bar{v})\left(x^*,t^*\right) &= d\left[\int_{\mathbb{R}} J\left(x^*-y\right)\bar{v}\left(y,t^*\right)\mathrm{d}y-\bar{v}\left(x^*,t^*\right)\right] \\ &\ge d\left[\bar{v}\left(x^*,t^*\right)-\bar{v}\left(x^*,t^*\right)\right] \\ &= 0. \end{aligned}$$

Therefore,

$$\int_{\mathbb{R}} J\left(x^*-y\right)\left(\bar{v}\left(y,t^*\right)-\bar{v}\left(x^*,t^*\right)\right)\mathrm{d}y=0.$$

Then, by (J4), for all $y\in(-\varrho+x^*,\varrho+x^*)$, we have that

$$\bar{v}\left(y,t^*\right)=\bar{v}\left(x^*,t^*\right). \tag{2.8}$$

By the same arguments, for an arbitrary $x_1\in\partial\left(x^*-\varrho,x^*+\varrho\right)$, we obtain (2.8), for all $y\in(x_1-\varrho,x_1+\varrho)$. Hence, (2.8) holds for all $y\in(x^*-2\varrho,x^*+2\varrho)$. As a result, (2.8) holds, for all $y\in\mathbb{R}$. Therefore, $\bar{v}\left(\cdot,t^*\right)$ is a constant, i.e.

$$\bar{v}\left(x,t^*\right)=\bar{v}\left(x^*,t^*\right)=0,\quad x\in\mathbb{R}.$$

Hence, by (2.2) for $x\in\mathbb{R}$,

$$\begin{aligned} -\bar{v}(x,0) &= \bar{v}\left(x,t^*\right)-\bar{v}(x,0) \\ &\ge d\int_0^{t^*}\int_{\mathbb{R}} J(x-y)\bar{v}(y,t)\mathrm{d}y\mathrm{d}t+\int_0^{t^*}\left(K-\left\|\bar{K}_0\right\|_X-d\right)\bar{v}(x,t)\mathrm{d}t \\ &= d\int_0^{t^*}\int_{\mathbb{R}} J(x-y)\bar{v}(y,t)\mathrm{d}y\mathrm{d}t+d\int_0^{t^*}\bar{v}(x,t)\mathrm{d}t\ge 0. \end{aligned}$$

This implies that $\bar{u}(x,0)\equiv 0$ for $x\in\mathbb{R}$, which is a contradiction. Therefore, $\bar{u}(x,t)>0$ on $\mathbb{R}\times(0,t_0]$. Thus $u(x,t)>0$ on $\mathbb{R}\times(0,t_0]$. This completes the proof. □

Jin and Zhao [23] gave the comparison principle of a scalar periodic logistic equation with nonlocal diffusion, see Corollary 2.2.

Corollary 2.2. ([23]) *Let $\bar{w}(x,t)$ and $\underline{w}(x,t)$ be upper and lower solutions of the following periodic equation:*

$$\frac{\partial w}{\partial t}(x,t) = d(t)\int_{\mathbb{R}} J(x-y)w(y,t)\mathrm{d}y + F(w(x,t),t), \quad x \in \mathbb{R}, t > 0,$$

with $\bar{w}(\cdot,t), \underline{w}(\cdot,t) \in X_s$ for all $t > 0$. Here the two continuous functions F and d are ω-periodic in t for some $\omega > 0$, $d(t) \geq 0$ and $d(t) \not\equiv 0$, kernel J satisfies (J1)-(J2) *and the initial data $\bar{w}(\cdot,0), \underline{w}(\cdot,0) \in X_s$ admit a nonempty compact support. Moreover, assume that there exists $L > 0$ such that $|F(\bar{w},t) - F(\underline{w},t)| \leq L|\bar{w} - \underline{w}|$ for all $t \geq 0$. If $\bar{w}(\cdot,0) \geq \underline{w}(\cdot,0)$, then $\bar{w}(\cdot,t) \geq \underline{w}(\cdot,t)$ for all $t \geq 0$.*

For our purpose, we would like to present a comparison principle for the following equation with constant coefficients:

$$\begin{cases} \frac{\partial w}{\partial t}(x,t) = d(J * w - w) + rw(x,t)(s - w(x,t)), & x \in \mathbb{R}, t > 0, \\ w(x,0) = \chi(x), & x \in \mathbb{R}, \end{cases} \tag{2.9}$$

where kernel J satisfies (J1)-(J2) and the initial data $\chi \in X_s$ admits a nonempty compact support. The parameters satisfy $d > 0, r > 0$ and $s > 0$.

Proposition 2.3. *Let w be a solution of* (2.9) *with $w(\cdot,t) \in X_s$ for all $t > 0$ for a given $\chi \in X_s$. If $z(\cdot,t) \in X_s$ and $z(x,t)$ satisfies*

$$\begin{cases} \frac{\partial z}{\partial t}(x,t) \geq d(J * z - z) + rz(x,t)(s - z(x,t)), & x \in \mathbb{R}, t > 0, \\ z(x,0) \geq \chi(x), & x \in \mathbb{R}, \end{cases}$$

then $z(x,t) \geq w(x,t)$ for all $x \in \mathbb{R}$, $t > 0$. Similar result holds for the reverse inequality.

Kao, Lou and Shen [24] gave the following comparison principle for nonlocal reaction-diffusion equation.

Corollary 2.4. ([24]) *Let $p_1, p_2, \cdots, p_N$ be given positive constants and*

$$Y_P = \left\{ v \in C\left(\mathbb{R}^N\right) \mid v\left(x_1, \cdots, x_{n-1}, x_n + p_n, x_{n+1}, \cdots, x_N\right) \right.$$
$$\left. = v\left(x_1, \cdots, x_{n-1}, x_n, x_{n+1}, \cdots, x_N\right), n = 1, 2, \cdots, N \right\}$$

with the norm $\|v\|_{Y_P} = \sup_{x \in \mathbb{R}^N} |v(x)|$. Assume that $\bar{v}(x,t)$ and $\underline{v}(x,t) \in Y_P$ are super-solution and sub-solution of

$$\frac{\partial v}{\partial t} = \int_{\mathbb{R}} \frac{1}{\delta^N} J\left(\frac{x-y}{\delta}\right) v \mathrm{d}y - v + vg(x,v), \quad (x,t) \in \mathbb{R}^N \times [0,\infty),$$

with $\bar{v}(x,0) = \bar{v}_0(x) \in Y_P$ and $\underline{v}(x,0) = \underline{v}_0(x) \in Y_P$, respectively. Here $\delta > 0$, $J(\cdot) \in C^\infty\left(\mathbb{R}^N\right)$ is defined by

$$J(x) = \begin{cases} C\exp\left(\frac{1}{|x|^2 - 1}\right) & \textit{for} \quad |x| < 1, \\ 0 \quad \textit{for} \quad |x| \geq 1, \end{cases}$$

where $C > 0$ is chosen such that $\int_{\mathbb{R}^N} J(x)\mathrm{d}x = 1$. Moreover, g is a smooth function and satisfies $g(x_1, \cdots, x_{n-1}, x_n + p_n, x_{n+1}, \cdots, x_N, v) = g(x_1, \cdots, x_{n-1}, x_n, x_{n+1}, \cdots, x_N, v)(n = 1, 2, \cdots, N)$, $g(x, 0) > 0$ and $g_v(x, v) := \frac{\partial g(x,v)}{\partial v} < 0$ for $x \in \mathbb{R}^N$ and $v \geq 0$, and $g(x, v) < 0$ for $x \in \mathbb{R}^N$ and $v > 1$. If $\underline{v}_0 \leq \bar{v}_0$, then $\underline{v}(\cdot, t) \leq \bar{v}(\cdot, t)$ for $0 \leq t < \infty$. Moreover $\underline{v}(\cdot, t) < \bar{v}(\cdot, t)$ if $\underline{v}_0 \neq \bar{v}_0$.

2.2. Some results for spreading speed and eigenvalue

In this subsection, we mainly recall some results for spreading speed and eigenvalue. First, according to the research of [23], we obtain that system (2.9) has the following spreading speed.

Proposition 2.5. *Let w be a solution of (2.9) with $w(\cdot, t) \in X_s$ for all $t > 0$ for a given $\chi \in X_s$ and $\bar{c} := \inf_{0<\lambda<+\infty} \frac{d[\int_{\mathbb{R}} J(x)\mathrm{e}^{\lambda x}\mathrm{d}x - 1] + rs}{\lambda} > 0$. Then the following statements are valid.*

(i) For any $c > \bar{c}$, if χ has a nonempty compact support, then

$$\lim_{t\to\infty} \sup_{|x|>ct} w(x, t) = 0;$$

(ii) For any $0 < c < \bar{c}$, if $\chi(\cdot) \not\equiv 0$, then

$$\liminf_{t\to\infty} \inf_{|x|<ct} w(x, t) = s.$$

Next, we consider the wave profile problem. The following lemma gives the existence of traveling wave solutions when the wave speed $c > c^*$.

Lemma 2.6. ([9], [55]) *Assume that $J \geq 0$ is even, compactly supported and $\int_{\mathbb{R}} J(x)\mathrm{d}x = 1$. Consider the system*

$$\begin{cases} J * u - u + cu' + f(u) = 0, \\ u(-\infty) = 1, \quad u(+\infty) = 0, \end{cases} \tag{2.10}$$

where the smooth function f satisfies

$$f(0) = f(1) = 0,\ f(s) > 0 \text{ for } s \in (0, 1),\ f'(0) > 0 \quad \text{and} \quad f'(s) < f'(0) \text{ for } s \in (0, 1).$$

Then for any $c \geq c^* := \inf_{0<\lambda<+\infty} \frac{\int_{\mathbb{R}} J(y)\mathrm{e}^{\lambda y}\mathrm{d}y - 1 + f'(0)}{\lambda} > 0$, system (2.10) has a unique (up to a shift of origin) smooth and monotonically decreasing solution $u(\xi), \xi := x - ct \in \mathbb{R}$.

Now we consider the eigenvalue problem

$$\begin{cases} (J * u)(x) - u(x) = -\lambda u, \quad x \in \Omega, \\ u(x) = 0, \quad x \in \mathbb{R}\backslash\Omega, \end{cases} \tag{2.11}$$

where $\Omega \subset \mathbb{R}$ is a bounded interval.

Lemma 2.7. ([19, Theorem 2.1]) *Assume that the kernel $J \in C(\mathbb{R})$, compactly supported and satisfies* (J1). *Then the problem* (2.11) *admits an eigenvalue $\lambda_1(\Omega)$ associated with a positive eigenfunction $\phi_1 \in C(\overline{\Omega})$. Moreover, it is simple and unique, and satisfies $0 < \lambda_1(\Omega) < 1$.*

Lemma 2.8. ([19, Theorem 1.4]) *For the principal eigenvalue $\lambda_1(\Omega)$ of problem* (2.11)*, we have $\lambda_1(\Omega) \to 1$ as $|\Omega| \to 0$, and $\lambda_1(\Omega_n) \to 0$ as $\Omega_n \to \mathbb{R}$.*

Define

$$c_U := \inf_{0<\lambda<+\infty} \frac{d_1\left[\int_{\mathbb{R}} J_1(x)\mathrm{e}^{-\lambda x}\mathrm{d}x - 1\right] + r_1}{\lambda}, \tag{2.12}$$

$$c_V := \inf_{0<\lambda<+\infty} \frac{d_2\left[\int_{\mathbb{R}} J_2(x)\mathrm{e}^{-\lambda x}\mathrm{d}x - 1\right] + r_2}{\lambda}, \tag{2.13}$$

and consider the functions

$$\Delta_1(\lambda, c) := d_1\left[\int_{\mathbb{R}} J_1(x)\mathrm{e}^{-\lambda x}\mathrm{d}x - 1\right] - c\lambda + r_1,$$

$$\Delta_2(\lambda, c) := d_2\left[\int_{\mathbb{R}} J_2(x)\mathrm{e}^{-\lambda x}\mathrm{d}x - 1\right] - c\lambda + r_2.$$

Then, it is easy to see that $\lambda \to \Delta_i(\lambda, c)$, $(i = 1, 2)$ is strictly convex with respect to λ for each given c. Moreover, due to $r_2 > 0$ and the definitions of c_U and c_V in (2.12) and (2.13) respectively, it enjoys the following properties.

Lemma 2.9. *Assume that the kernel J_1 (or J_2) satisfies* (J1)-(J2)*, then the following statements hold.*

- **(i)** *If $c > c_U$ (or c_V), then the equation $\Delta_1(\lambda, c) = 0$ (or $\Delta_2(\lambda, c) = 0$) has two positive roots $\lambda_1(c)$, $\lambda_2(c)$, and $0 < \lambda_1(c) < \lambda_2(c) < +\infty$. Moreover, $\Delta_1(\lambda, c) < 0$ (or $\Delta_2(\lambda, c) < 0$), if $\lambda \in (\lambda_1(c), \lambda_2(c))$. $\Delta_1(\lambda, c) > 0$ (or $\Delta_2(\lambda, c) > 0$), if $\lambda \in (0, \lambda_1(c)) \cup (\lambda_2(c), +\infty)$.*
- **(ii)** *If $0 < c < c_U$ (or c_V), then $\Delta_1(\lambda, c) > 0$ (or $\Delta_2(\lambda, c) > 0$) for all $\lambda \in (0, +\infty)$.*
- **(iii)** *If $c = c_U$ (or c_V)> 0, then $\Delta_1(\lambda, c_U) \geq 0$ (or $\Delta_2(\lambda, c) \geq 0$) for any $\lambda \in (0, +\infty)$ and $\Delta_1(\lambda, c_U) = 0$ (or $\Delta_2(\lambda, c) = 0$) has a unique root λ^*.*

3. Slow predator case

In this section, we consider the case when the predator spreads slower than the prey and establish the following theorem.

Theorem 3.1. *(Slow predator) Assume that* (J1), (J3)-(J4) *and* (H1) *hold. Let u_0, v_0 be two given nontrivial compactly supported functions such that $(u_0, v_0) \in H$. In addition, we assume that $d_1 > r_1 + b\hat{r}_2 + \frac{b}{2} + \beta b\hat{r}_2$ and $d_2 > \beta b - a + \beta b\hat{r}_2 + \frac{b}{2}$.*

If $c_V < c_U$, then the solution $(U, V) \equiv (U(x,t), V(x,t))$ of (1.1) *with initial data (u_0, v_0) satisfies the following statements.*

(i) For any $c > c_U$, then

$$\lim_{t\to\infty} \sup_{|x|>ct} U(x,t) = 0.$$

(ii) For any $c_V < c_1 < c_2 < c_U$ and each $c > c_V$, one has:

$$\lim_{t\to\infty} \sup_{c_1 t<|x|<c_2 t} |1 - U(x,t)| + \sup_{|x|>ct} V(x,t) = 0.$$

(iii) There exists $\varepsilon > 0$ such that for each $c \in [0, c_V)$, one has

$$\liminf_{t\to\infty} \inf_{|x|\le ct} V(x,t) \ge \varepsilon,$$

$$\limsup_{t\to\infty} \sup_{|x|\le ct} U(x,t) \le 1 - \varepsilon \text{ and } \liminf_{t\to\infty} \inf_{|x|\le ct} U(x,t) \ge \varepsilon.$$

We divide the proof of Theorem 3.1 into four subsections. In subsection 3.1, we will show that the prey U cannot spread faster than c_U. In subsection 3.2, we will prove that U converges to 1 and V is close to 0 as $t \to \infty$ in the case $c > c_V$ and $c_V < c_1 < c_2 < c_U$. In the case $0 \le c < c_V$, we will prove that U and V remain uniformly positive in subsection 3.3. In subsection 3.4, we complete the proof of Theorem 3.1 by giving the proof of Claim* introduced in subsection 3.3.

In the rest of this section, we always assume $c_V < c_U$, (J1), (J3)-(J4) and (H1) hold.

3.1. Spreading of U for $c > c_U$

Proposition 3.2. *If $c > c_U$, then*

$$\lim_{t\to\infty} \sup_{|x|>ct} U(x,t) = 0. \tag{3.1}$$

Proof. Since $U \ge 0$ and $V \ge 0$, the function U satisfies

$$\frac{\partial U}{\partial t}(x,t) \le d_1(J_1 * U - U)(x,t) + r_1 U(x,t)(1 - U(x,t)).$$

Applying Proposition 2.3, we have $U(x,t) \le \overline{U}(x,t)$ for all $(x,t) \in \mathbb{R} \times [0,\infty)$, where the function $\overline{U}$ is the solution of the following system

$$\begin{cases} \frac{\partial \overline{U}}{\partial t}(x,t) = d_1(J_1 * \overline{U} - \overline{U})(x,t) + r_1 \overline{U}(x,t)(1 - \overline{U}(x,t)), \\ \overline{U}(x,0) = u_0(x). \end{cases}$$

According to the comparison principle and Proposition 2.5, we get that

$$\forall c > c_U, \ \lim_{t\to\infty} \sup_{|x|>ct} U(x,t) \le \lim_{t\to\infty} \sup_{|x|>ct} \overline{U}(x,t) = 0. \tag{3.2}$$

Since U is nonnegative, thus we prove (3.1). □

3.2. Spreading of U and V for $c > c_V$ and $c_V < c_1 < c_2 < c_U$

Proposition 3.3. *For any $c_V < c_1 < c_2 < c_U$ and each $c > c_V$ one has:*

$$\lim_{t\to\infty} \sup_{c_1 t<|x|<c_2 t} |1-U(x,t)| + \sup_{|x|>ct} V(x,t) = 0.$$

Proof. Since $U \le 1$ and the function $\frac{\beta b U}{1+\alpha U}$ increases monotonically with respect to U, we obtain that

$$\frac{\partial V}{\partial t}(x,t) \le d_2(J_2 * V - V)(x,t) + V(x,t)\left[\frac{\beta b}{1+\alpha} - a - aV(x,t)\right].$$

By Proposition 2.3, we have

$$V(x,t) \le \overline{V}(x,t), \text{ for all } (x,t) \in \mathbb{R}\times[0,\infty), \tag{3.3}$$

where the function $\overline{V}$ is the solution of the following equations:

$$\begin{cases} \frac{\partial \overline{V}}{\partial t}(x,t) = d_2(J_2 * \overline{V} - \overline{V})(x,t) + a\overline{V}(x,t)(\hat{r}_2 - \overline{V}(x,t)), \\ \overline{V}(x,0) = v_0(x), \end{cases} \tag{3.4}$$

where $\hat{r}_2 = \frac{r_2}{a} = \frac{1}{a}(\frac{\beta b}{1+\alpha} - a)$. It follows from Proposition 2.5 that for all $c > c_V$

$$\lim_{t\to\infty} \sup_{|x|>ct} V(x,t) \le \lim_{t\to\infty} \sup_{|x|>ct} \overline{V}(x,t) = 0.$$

Since $V \ge 0$, we have for all $c > c_V$,

$$\lim_{t\to\infty} \sup_{|x|>ct} V(x,t) = 0. \tag{3.5}$$

Next, we only need to prove that for any $c_V < c_1 < c_2 < c_U$,

$$\lim_{t\to\infty} \sup_{c_1 t<|x|<c_2 t} |1-U(x,t)| = 0. \tag{3.6}$$

To prove (3.6), we give the following lemma.

Lemma 3.4. *Let $d_1 > r_1 + b\hat{r}_2 + \frac{b}{2} + \beta b\hat{r}_2$ and $d_2 > \beta b - a + \beta b\hat{r}_2 + \frac{b}{2}$. For any $c \in (c_V, c_U)$, we have*

$$\lim_{t\to+\infty} U(x+ct,t) = 1$$

uniformly on every compact subset of $\mathbb{R}$.

Proof. First, we choose $c \in (c_V, c_U)$. According to the definition of $\overline{V}(x,t)$ and (3.5), there is some $X_\varepsilon > 0$ such that

$$V(x,t) \leq \varepsilon, \text{ for all } (x,t) \text{ such that } |x| \geq X_\varepsilon + ct.$$

We divide the proof into three steps.

Step 1. Taking $c' \in (c, c_U)$, we claim that there exist $l_2 > 0$, $x_1 \in \mathbb{R}$ and $\eta_1 > 0$ such that

$$\liminf_{t\to\infty} \inf_{x\in(-l_2,l_2)} U\left(x + ct + x_1, \frac{ct}{c'}\right) \geq \eta_1. \tag{3.7}$$

Let $l_2 > 0$, $\eta > 0$ and $\alpha_1 > 0$. We take $x_1 = X_\varepsilon + 2l_2$. Define

$$\underline{U}(x,t) := \eta e^{-\alpha_1(x-c't)}\psi_{2l_2}(x - c't - x_1), \tag{3.8}$$

where

$$\psi_{2l_2}(x) = \begin{cases} \cos\left(\frac{\pi x}{4l_2}\right), & x \in (-2l_2, 2l_2), \\ 0, & x \in \mathbb{R} \setminus (-2l_2, 2l_2). \end{cases} \tag{3.9}$$

Next, we prove that $\underline{U}$ is a sub-solution of the U-equation in system (1.1). Indeed, we have $\underline{U}(x,t) > 0$ for $-2l_2 < x - c't - x_1 < 2l_2$ and $t \in [0, +\infty)$. Consider the following operators for $(x,t) \in (-2l_2 + c't + x_1, 2l_2 + c't + x_1) \times [0, +\infty)$

$$\begin{aligned} Q[W](x,t) :=& -\partial_t W(x,t) + d_1 \left[\int_{\mathbb{R}} J_1(x-y)W(y,t)\mathrm{d}y - W(x,t)\right] \\ &+ W(x,t)\left[r_1(1 - W(x,t)) - \frac{b\varepsilon}{1+\alpha W(x,t)}\right], \end{aligned}$$

and

$$L[W](x,t) := -\partial_t W(x,t) + d_1 \int_{\mathbb{R}} J_1(x-y)W(y,t)\mathrm{d}y + mW(x,t), \tag{3.10}$$

where m will be determined later. We claim that for l_2 large enough

$$\begin{aligned} J_1 * \underline{U}(x,t) =& \eta \int_{-2l_2+c't+x_1}^{2l_2+c't+x_1} J_1(x-y)\mathrm{e}^{-\alpha_1(y-c't)} \cos\left(\frac{\pi(y - c't - x_1)}{4l_2}\right)\mathrm{d}y \\ \geq& \eta \int_{-\infty}^{\infty} J_1(x-y)\mathrm{e}^{-\alpha_1(y-c't)} \cos\left(\frac{\pi(y - c't - x_1)}{4l_2}\right)\mathrm{d}y. \end{aligned} \tag{3.11}$$

Indeed, due to anyhow $J_1 * \underline{U}(x,t) \geq 0$, we take without loss of generality that $x \in (-2l_2 + c't + x_1, 2l_2 + c't + x_1)$. In order to make (3.11) hold we have to show that for $y \in \mathbb{R} \setminus (-2l_2 + c't +$

$x_1, 2l_2 + c't + x_1)$ either $\cos\left(\frac{\pi(y-c't-x_1)}{4l_2}\right) \le 0$ or $J_1(x-y) = 0$. We assume J_1 has a compact support, then there exists $\tilde{B}$ such that supp $J_1 \subset [-\tilde{B}, \tilde{B}]$. If $x \in (-2l_2 + c't + x_1, 2l_2 + c't + x_1)$ and $|x-y| \le \tilde{B}$, then $y \in (-2l_2 + c't + x_1 - \tilde{B}, 2l_2 + c't + x_1 + \tilde{B})$ which implies that $y - c't - x_1 \in (-2l_2 - \tilde{B}, 2l_2 + \tilde{B}) \subset (-6l_2, 6l_2)$ when $\tilde{B} \le 4l_2$. Thus, we obtain that $\cos\left(\frac{\pi(y-c't-x_1)}{4l_2}\right) \le 0$ for $y - c't - x_1 \in [-6l_2, -2l_2] \cup [2l_2, 6l_2]$. Moreover, since supp $J_1 \subset [-\tilde{B}, \tilde{B}] \subset (-4l_2, 4l_2)$ when $\tilde{B} < 4l_2$, we obtain that $J_1(x-y) = 0$ for $y \in \mathbb{R} \setminus (-6l_2 + c't + x_1, 6l_2 + c't + x_1)$. Thus, we deduce that

$$
\begin{aligned}
&\eta \int_{-\infty}^{\infty} J_1(x-y) \mathrm{e}^{-\alpha_1(y-c't)} \cos\left(\frac{\pi(y-c't-x_1)}{4l_2}\right) \mathrm{d}y \\
=&\Bigg(\int_{-\infty}^{-6l_2+c't+x_1} + \int_{-6l_2+c't+x_1}^{-2l_2+c't+x_1} + \int_{-2l_2+c't+x_1}^{2l_2+c't+x_1} + \int_{2l_2+c't+x_1}^{6l_2+c't+x_1} \\
&+ \int_{6l_2+c't+x_1}^{\infty} \Bigg) J_1(x-y) \eta \mathrm{e}^{-\alpha_1(y-c't)} \cos\left(\frac{\pi(y-c't-x_1)}{4l_2}\right) \mathrm{d}y \\
\le& \eta \int_{-2l_2+c't+x_1}^{2l_2+c't+x_1} J_1(x-y) \mathrm{e}^{-\alpha_1(y-c't)} \cos\left(\frac{\pi(y-c't-x_1)}{4l_2}\right) \mathrm{d}y.
\end{aligned}
$$

Substituting (3.8) into (3.10) and using (3.11), we obtain that

$$
\begin{aligned}
L[\underline{U}](x,t) \ge& c' \left[-\alpha_1 \eta e^{-\alpha_1(x-c't)} \psi_{2l_2}(x-c't-x_1) - \frac{\pi\eta}{4l_2} \mathrm{e}^{-\alpha_1(x-c't)} \sin\left(\frac{\pi(x-c't-x_1)}{4l_2}\right) \right] \\
&+ m\underline{U}(x,t) + d_1 \eta \int_{-\infty}^{\infty} J_1(x-y) \mathrm{e}^{-\alpha_1(y-c't)} \cos\left(\frac{\pi(y-c't-x_1)}{4l_2}\right) \mathrm{d}y \\
=& \left[-c'\alpha_1 + m + d_1 \int_{\mathbb{R}} \mathrm{e}^{\alpha_1 y} J_1(y) \cos\left(\frac{\pi y}{4l_2}\right) \mathrm{d}y \right] \underline{U}(x,t) \\
&+ \left[-\frac{\pi}{4l_2} c' + d_1 \int_{\mathbb{R}} \mathrm{e}^{\alpha_1 y} J_1(y) \sin\left(\frac{\pi y}{4l_2}\right) \mathrm{d}y \right] \eta \mathrm{e}^{-\alpha_1(x-c't)} \sin\left(\frac{\pi(x-c't-x_1)}{4l_2}\right).
\end{aligned}
$$

Therefore, $L_1\left[\underline{U}\right](x,t) > 0$ on $x - c't - x_1 \in (-2l_2, 2l_2)$ and $t \in [0, +\infty)$ if the following two conditions are satisfied:

$$
c' < \frac{1}{\alpha_1} \left[m + d_1 \int_{\mathbb{R}} \mathrm{e}^{\alpha_1 y} J_1(y) \cos\left(\frac{\pi y}{4l_2}\right) \mathrm{d}y \right] =: \mathcal{A}_m(\alpha_1, l_2), \tag{3.12}
$$

$$c' = \frac{4l_2 d_1}{\pi}\left[\int_{\mathbb{R}} e^{\alpha_1 y} J_1(y) \sin\left(\frac{\pi y}{4l_2}\right) dy\right] =: \mathcal{B}(\alpha_1, l_2). \tag{3.13}$$

We first show some properties of the functions $\mathcal{A}_m$ and $\mathcal{B}$. As $l_2 \to \infty$, we have locally uniform convergence of

$$\mathcal{A}_m(\alpha_1, l_2) \to A_m(\alpha_1) = \frac{m + d_1 \int_{\mathbb{R}} e^{\alpha_1 y} J_1(y) dy}{\alpha_1}, \quad \mathcal{B}(\alpha_1, l_2) \to B(\alpha_1) := d_1 \int_{\mathbb{R}} y e^{\alpha_1 y} J_1(y) dy.$$

Differentiation yields

$$A_m'(\alpha_1) = (B(\alpha_1) - A_m(\alpha_1))/\alpha_1, \tag{3.14}$$
$$B'(\alpha_1) = d_1 \int_{\mathbb{R}} J(y) e^{\alpha_1 y} y^2 dy > 0.$$

It follows from the properties of the function $A_m(\alpha_1)$ that it achieves infimum. Then, there exists $\alpha_1^* > 0$ such that $A_m(\alpha_1^*) = \inf_{\alpha_1 > 0} A_m(\alpha_1)$. By the definition of α_1^* and (3.14), we have that $B(\alpha_1^*) = A_m(\alpha_1^*)$. Since B is an increasing function, thus $B(\alpha_1) < B(\alpha_1^*)$ for $0 < \alpha_1 < \alpha_1^*$. Then we have

$$A_m(\alpha_1) > A_m(\alpha_1^*) = B(\alpha_1^*) > B(\alpha_1) \text{ for } 0 < \alpha_1 < \alpha_1^*. \tag{3.15}$$

In addition, we define $c^* := A_{m^*}(\alpha_1^*)$ with $m^* = r_1 - d_1 - b\varepsilon - r_1\delta$ and $r_1 - b\varepsilon - r_1\delta > 0$ for small enough constant $\delta > 0$. We can choose $m < m^*$ such that $c' < A_m(\alpha_1^*) < A_{m^*}(\alpha_1^*) = c^* < c_U$. Note that $B(0) < c^*$ and $B(0) = 0$, then we have $c' > B(0)$. Therefore, combined with (3.15), we can choose $\hat{c}_1, \hat{c}_2, \delta_1, l_2 > 0$ such that

$$B(\hat{c}_1) + \delta_1 < c' < B(\hat{c}_2) - \delta_1 \text{ and } |\mathcal{B}(\alpha_1, l_2) - B(\alpha_1)| < \delta_1.$$

By the continuity of $\mathcal{B}(\alpha_1, l_2)$ and $B(\alpha_1)$, there exists some $\alpha_1(l_2)$ such that $\mathcal{B}(\alpha_1(l_2), l_2) = c'$ for all large enough l_2. We can also choose l_2 large enough such that $\mathcal{A}_m(\alpha_1(l_2), l_2) > c'$. Thus, we prove that (3.12) and (3.13) hold true.

Therefore we have $L_1[\underline{U}](x,t) > 0$ for $x - c't - x_1 \in (-2l_2, 2l_2)$ and $t \in [0, +\infty)$. Note that

$$r_1 W(x,t)(1 - W(x,t)) - \frac{b\varepsilon W(x,t)}{1 + \alpha W(x,t)} \geq (r_1 - r_1\delta - b\varepsilon) W(x,t) \text{ for } 0 \leq W(x,t) \leq \delta.$$

Note that $m < r_1 - d_1 - b\varepsilon - r_1\delta$ implies that

$$Q\left[\underline{U}\right](x,t) > L_1\left[\underline{U}\right](x,t) > 0 \text{ for } x - c't - x_1 \in (-2l_2, 2l_2) \text{ and } t \in [0, \infty),$$

namely, for $x - c't - x_1 \in (-2l_2, 2l_2)$ and $t \in [0, \infty)$, we have that

$$\partial_t \underline{U}(x,t) - d_1(J_1 * \underline{U}(x,t) - \underline{U}(x,t)) - \underline{U}(x,t)\left(r_1 - r_1\underline{U}(x,t) - \frac{b\varepsilon}{1 + \alpha \underline{U}(x,t)}\right) < 0.$$

Furthermore since supp $\psi_{2l_2} \subset [-2l_2, 2l_2]$, we have for $(x,t) \in \mathbb{R} \times [0, +\infty)$

$$\partial_t \underline{U}(x,t) - d_1(J_1 * \underline{U}(x,t) - \underline{U}(x,t)) - \underline{U}(x,t)\left(r_1 - r_1\underline{U}(x,t) - \frac{b\varepsilon}{1+\alpha\underline{U}(x,t)}\right) \le 0.$$

Since function $u_0(x)$ is not trivial, one can make use of comparison principle to get that $0 < U(x,t) < 1$ for all $x \in \mathbb{R}$ and $t > 0$. Thus, for any fixed $t_0 > 0$, we reduce η such that for $x \in \mathbb{R}$ and $t \ge t_0$, $U(x,t_0) \ge \underline{U}(x,t_0)$. Then, by comparison principle, we obtain that

$$U\left(x, \frac{ct}{c'}\right) \ge \underline{U}\left(x, \frac{ct}{c'}\right) = \eta e^{-\alpha_1(x-ct)}\psi_{2l_2}(x-ct-x_1) = \underline{U}(x-ct, 0).$$

The claim (3.7) can be obtained by taking

$$\eta_1 = \eta e^{-\alpha_1(X_\varepsilon + 3l_2)} \min_{x\in[-l_2,l_2]} \psi_{2l_2}(x).$$

Step 2. We now claim that there exist $l_2 > 0$, $\eta_2 > 0$ and $x_3 \in \mathbb{R}$ such that

$$\liminf_{t\to\infty} \inf_{x\in(-\frac{l_2}{2},\frac{l_2}{2}),t'\in[\frac{ct}{c'},t]} U(x+ct+x_3, t') \ge \eta_2. \tag{3.16}$$

Define

$$\underline{W}(x) := \eta'\hat{\psi}_{l_2}(x - x_1),$$

where $\eta' > 0$ will be determined later and $\hat{\psi}_{l_2}$ is the eigenvalue function of the following eigenvalue problem

$$\begin{cases} (J_1 * \hat{\psi}_{l_2})(x) - \hat{\psi}_{l_2}(x) = -\hat{\lambda}_{l_2}\hat{\psi}_{l_2}(x), & x \in (-l_2, l_2), \\ \hat{\psi}_{l_2}(x) = 0, \quad x \in \mathbb{R}\backslash(-l_2, l_2), \\ \|\hat{\psi}_{l_2}\|_\infty = 1. \end{cases}$$

Then, for any $t' \in \left[\frac{ct}{c'}, t\right]$, we have

$$\begin{aligned} &\frac{\partial \underline{W}}{\partial t'}(x) - d_1(J_1 * \underline{W} - \underline{W})(x) - r_1\underline{W}(x)\left(1 - \underline{W}(x) - \frac{bV(x,t')}{r_1(1+\alpha\underline{W}(x))}\right) \\ \le &\frac{\partial \underline{W}}{\partial t'}(x) - d_1(J_1 * \underline{W} - \underline{W})(x) - r_1\underline{W}(x)\left(1 - \underline{W}(x) - \frac{b\varepsilon}{r_1}\right) \\ = &- \underline{W}(x)\left[-d_1\hat{\lambda}_{l_2} + r_1\left(1 - \underline{W}(x) - \frac{b\varepsilon}{r_1}\right)\right] \\ \le &0. \end{aligned}$$

The last inequality is obtained by assuming $r_1 - d_1\hat{\lambda}_{l_2} - b\varepsilon > 0$ (increasing l_2 if necessary) and taking $\eta' < 1 - \frac{d_1}{r_1}\hat{\lambda}_{l_2} - \frac{b\varepsilon}{r_1}$. It can be deduced from (3.7) that for t large enough and $x \in (-l_2, l_2)$, one has $U(x+ct+x_1, \frac{ct}{c'}) \geq \eta_1$. Thus, we can reduce η' such that $U(x+ct+x_1, \frac{ct}{c'}) \geq \underline{W}(x+x_1) = \eta'\hat{\psi}_{l_2}$. Then, by comparison principle,

$$U \geq \underline{W}, \quad \forall x \in \mathbb{R} \text{ and } t' \in \left[\frac{ct}{c'}, t\right].$$

Therefore, by taking $x_3 = x_1$ and $\eta_2 = \eta' \min_{x\in[-\frac{l_2}{2},\frac{l_2}{2}]} \hat{\psi}_{l_2}(x)$, we know that (3.16) holds.

Step 3. We complete the proof of Lemma 3.4. Let $\{t_n\}_{n\in\mathbb{Z}}$ be such that $t_n \to \infty$, as $n \to \infty$. Define

$$\begin{cases} U_n(x,t) = U\left(x+ct_n, t+t_n\right), \\ V_n(x,t) = V\left(x+ct_n, t+t_n\right), \end{cases}$$

for $(x,t) \in \mathbb{R} \times [-t_n, +\infty)$. It is clearly that $(U_n(x,t), V_n(x,t))$ satisfies

$$\begin{cases} \frac{\partial U_n}{\partial t}(x,t) = d_1\left[(J_1 * U_n)(x,t) - U_n(x,t)\right] + U_n(x,t)\left[r_1(1-U_n(x,t)) - \frac{bV_n(x,t)}{1+\alpha U_n(x,t)}\right], \\ \frac{\partial V_n}{\partial t}(x,t) = d_2\left[(J_2 * V_n)(x,t) - V_n(x,t)\right] + V_n(x,t)\left[\frac{\beta b U_n(x,t)}{1+\alpha U_n(x,t)} - a - aV_n(x,t)\right], \\ U_n\left(x,-t_n\right) = U\left(x+ct_n, 0\right), V_n\left(x,-t_n\right) = V\left(x+ct_n,0\right). \end{cases}$$

Next, we will give some priori estimates of $(U_n(x,t), V_n(x,t))$ uniformly in n, which allow us to reach to the limit as $n \to +\infty$. Since $0 \leq U(x,0) = u_0 \leq 1$, $0 \leq V(x,0) = v_0 \leq \hat{r}_2$, we have $0 \leq U_n(x,-t_n) \leq 1, 0 \leq V_n(x,-t_n) \leq \hat{r}_2$. Hence, $0 \leq U_n(x,t) \leq 1, 0 \leq V_n(x,t) \leq \hat{r}_2$. Thus, there exist positive constants $D_i, i = 1,2,\ldots,4$, such that for $(x,t) \in \mathbb{R} \times [-t_n, +\infty)$ and $n \in \mathbb{Z}$,

$$\begin{vmatrix} \frac{\partial U_n}{\partial t} \end{vmatrix} \leq d_1\left|J_1 * U_n\right| + d_1\left|U_n\right| + \left|U_n\right|\left[r_1(1+\left|U_n\right|) + \frac{b\left|V_n\right|}{\left|1+\alpha U_n\right|}\right] \leq 2d_1 + (2r_1 + b\hat{r}_2) =: D_1,$$
$$\begin{vmatrix} \frac{\partial V_n}{\partial t} \end{vmatrix} \leq d_2\left|J_2 * V_n\right| + d_2\left|V_n\right| + \left|V_n\right|\left[\frac{\beta b\left|U_n\right|}{\left|1+\alpha U_n\right|} + a + a\left|V_n\right|\right] \leq \hat{r}_2\left(2d_2 + \beta b + a + a\hat{r}_2\right) =: D_2,$$

and

$$\begin{aligned} \left|\frac{\partial^2 U_n}{\partial t^2}\right| &\leq d_1|J_1 * (U_n)_t)| + d_1|(U_n)_t| + |(U_n)_t|\left[r_1(1+|U_n|) + \frac{b|V_n|}{|1+\alpha U_n|}\right] \\ &\quad + |U_n|\left[r_1(1+\left|(U_n)_t\right|) + b\left|\left(\frac{V_n}{1+\alpha U_n}\right)_t\right|\right] \\ &\leq 2d_1D_1 + D_1(2r_1 + b\hat{r}_2) + r_1(1+D_1) + b[D_2 + \alpha D_2 + \alpha\hat{r}_2 D_1] =: D_3, \\ \left|\frac{\partial^2 V_n}{\partial t^2}\right| &\leq 2d_2D_2 + D_2\left(\beta b + a + a\hat{r}_2\right) + \hat{r}_2[\beta b D_1 + aD_2] =: D_4. \end{aligned}$$

For any $\gamma > 0$, define

$$\begin{cases} \mathcal{U}_{n,\gamma}(x,t) := U_n(x+\gamma,t) - U_n(x,t), \\ \mathcal{V}_{n,\gamma}(x,t) := V_n(x+\gamma,t) - V_n(x,t), \\ \tilde{J}_i(x) := J_i(x+\gamma) - J_i(x), i = 1,2. \end{cases}$$

Since J_i satisfies (J1) and (J3), then $J_i' \in L^1$, there exists $L_i > 0, i = 1,2$, such that

$$\int_{\mathbb{R}} \left|\tilde{J}_i(x-y)\right| \mathrm{d}y = \int_{\mathbb{R}} |J_i(x+\gamma-y) - J_i(x-y)| \,\mathrm{d}y$$
$$= |\gamma| \int_{\mathbb{R}} \left| \int_0^1 J_i'(x-y+\theta\gamma) d\theta \right| \mathrm{d}y$$
$$\le |\gamma| \int_0^1 \int_{\mathbb{R}} \left|J_i'(x-y+\theta\gamma)\right| \mathrm{d}y\mathrm{d}\theta \le L_i|\gamma|.$$

Hence, for any $\eta > 0$, there exists $\delta_i = \frac{\eta}{L_i} > 0$ such that $\int_{\mathbb{R}} |\tilde{J}_i(x-y)|\mathrm{d}y \le \eta$ provided that $|\gamma| \le \delta_i, x \in \mathbb{R}, i = 1, 2$. Then it can be verified that

$$\begin{aligned}
&\frac{\partial}{\partial t}\mathcal{U}_{n,\gamma}^2(x,t) = 2\mathcal{U}_{n,\gamma}(x,t)\frac{\partial \mathcal{U}_{n,\gamma}}{\partial t}(x,t) \\
&= 2\mathcal{U}_{n,\gamma}(x,t)\Big(d_1 \int_{\mathbb{R}} \tilde{J}_1(x-y)U_n(y,t)dy - (d_1-r_1)\mathcal{U}_{n,\gamma}(x,t) \\
&\quad - r_1\mathcal{U}_{n,\gamma}(x,t)(U_n(x+\gamma,t)+U_n(x,t)) - \frac{bV_n(x+\gamma,t)}{1+\alpha U_n(x+\gamma,t)}\mathcal{U}_{n,\gamma}(x,t) \\
&\quad + \frac{b\alpha U_n(x,t)V_n(x,t)\mathcal{U}_{n,\gamma}(x,t)}{[1+\alpha U_n(x+\gamma,t)][1+\alpha U_n(x,t)]} - \frac{bU_n(x,t)\mathcal{V}_{n,\gamma}(x,t)}{1+\alpha U_n(x+\gamma,t)}\Big) \\
&\le 2\mathcal{U}_{n,\gamma}(x,t)\Big(d_1 \int_{\mathbb{R}} \tilde{J}_1(x-y)U_n(y,t)dy - (d_1-r_1)\mathcal{U}_{n,\gamma}(x,t) \\
&\quad - r_1\mathcal{U}_{n,\gamma}(x,t)(U_n(x+\gamma,t)+U_n(x,t)) - \frac{bV_n(x+\gamma,t)}{1+\alpha U_n(x+\gamma,t)}\mathcal{U}_{n,\gamma}(x,t) \\
&\quad + \frac{b\alpha U_n(x,t)V_n(x,t)\mathcal{U}_{n,\gamma}(x,t)}{[1+\alpha U_n(x+\gamma,t)][1+\alpha U_n(x,t)]}\Big) \\
&\quad + \frac{bU_n(x,t)}{1+\alpha U_n(x+\gamma,t)}\Big(\mathcal{U}_{n,\gamma}^2(x,t) + \mathcal{V}_{n,\gamma}^2(x,t)\Big) \\
&\le 4d_1\eta - 2\left(d_1 - r_1 - b\hat{r}_2 - \frac{b}{2}\right)\mathcal{U}_{n,\gamma}^2(x,t) + b\mathcal{V}_{n,\gamma}^2(x,t),
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial t}\mathcal{V}_{n,\gamma}^2(x,t) = 2\mathcal{V}_{n,\gamma}(x,t)\frac{\partial \mathcal{V}_{n,\gamma}}{\partial t}(x,t) \\
&= 2\mathcal{V}_{n,\gamma}(x,t)\Big(d_2 \int_{\mathbb{R}} \tilde{J}_2(x-y)V_n(y,t)dy - (d_2+a)\mathcal{V}_{n,\gamma}(x,t) \\
&\quad - a(V_n(x+\gamma,t)+V_n(x,t))\mathcal{V}_{n,\gamma}(x,t) + \frac{\beta bU_n(x,t)}{1+\alpha U_n(x+\gamma,t)}\mathcal{V}_{n,\gamma}(x,t)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\beta b V_n(x+\gamma,t)}{1+\alpha U_n(x+\gamma,t)}\mathcal{U}_{n,\gamma}(x,t)-\frac{\beta b\alpha U_n(x,t)V_n(x,t)}{[1+\alpha U_n(x+\gamma,t)][1+\alpha U_n(x,t)]}\mathcal{U}_{n,\gamma}(x,t)\Big)\\
\le\,& 2\mathcal{V}_{n,\gamma}(x,t)\Big(d_2\int_{\mathbb{R}}\tilde{J}_2(x-y)V_n(y,t)dy-(d_2+a)\mathcal{V}_{n,\gamma}(x,t)\\
&+\frac{\beta b U_n(x,t)}{1+\alpha U_n(x+\gamma,t)}\mathcal{V}_{n,\gamma}(x,t)\Big)+\frac{\beta b V_n(x+\gamma,t)}{1+\alpha U_n(x+\gamma,t)}(\mathcal{U}^2_{n,\gamma}(x,t)+\mathcal{V}^2_{n,\gamma}(x,t))\\
&+\frac{\beta b\alpha U_n(x,t)V_n(x,t)}{[1+\alpha U_n(x+\gamma,t)][1+\alpha U_n(x,t)]}(\mathcal{U}^2_{n,\gamma}(x,t)+\mathcal{V}^2_{n,\gamma}(x,t))\\
\le\,& 4d_2\eta\hat{r}_2^2-2\left(d_2+a-\beta b-\beta b\hat{r}_2\right)\mathcal{V}^2_{n,\gamma}(x,t)+2\beta b\hat{r}_2\mathcal{U}^2_{n,\gamma}(x,t).
\end{aligned}
\tag{3.18}
$$

Adding the two inequalities (3.17) and (3.18), we deduce from the assumptions that $k_1 := d_1 - r_1 - b\hat{r}_2 - \frac{b}{2} - \beta b\hat{r}_2 > 0$ and $k_2 := d_2 - \beta b + a - \beta b\hat{r}_2 - \frac{b}{2} > 0$. Then we obtain that

$$
\begin{aligned}
&\frac{\partial}{\partial t}(\mathcal{U}^2_{n,\gamma}(x,t)+\mathcal{V}^2_{n,\gamma}(x,t))\\
&\le 4\left(d_1+d_2\hat{r}_2^2\right)\eta-2\left(d_1-r_1-b\hat{r}_2-\frac{b}{2}-\beta b\hat{r}_2\right)\mathcal{U}^2_{n,\gamma}(x,t)\\
&\quad -2\left(d_2+a-\beta b-\beta b\hat{r}_2-\frac{b}{2}\right)\mathcal{V}^2_{n,\gamma}(x,t)\\
&=4\left(d_1+d_2\hat{r}_2^2\right)\eta-2k_1\mathcal{U}^2_{n,\gamma}(x,t)-2k_2\mathcal{V}^2_{n,\gamma}(x,t).
\end{aligned}
\tag{3.19}
$$

Let $k=\min\{k_1,k_2\}$. Due to (3.19), we have

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\mathcal{U}^2_{n,\gamma}(x,t)+\mathcal{V}^2_{n,\gamma}(x,t)\right)\\
\le& 4\left(d_1+d_2\hat{r}_2^2\right)\eta-2k\left(\mathcal{U}^2_{n,\gamma}(x,t)+\mathcal{V}^2_{n,\gamma}(x,t)\right).
\end{aligned}
\tag{3.20}
$$

Multiplying both sides of (3.20) by $e^{2k(t-s)}$ and integrating from s to t, we have

$$
\begin{aligned}
&\left(\mathcal{U}^2_{n,\gamma}(x,t)+\mathcal{V}^2_{n,\gamma}(x,t)\right)\\
\le& e^{-2k(t-s)}\left(\mathcal{U}^2_{n,\gamma}(x,s)+\mathcal{V}^2_{n,\gamma}(x,s)\right)+4\left(d_1+d_2\hat{r}_2^2\right)\eta\int_s^t e^{-2k(t-\theta)}d\theta.
\end{aligned}
\tag{3.21}
$$

Taking $s=-t_n$ from (3.21), we get that

$$
\begin{aligned}
&\left(\mathcal{U}^2_{n,\gamma}(x,t)+\mathcal{V}^2_{n,\gamma}(x,t)\right)\\
\le& e^{-2k(t+t_n)}\left(\mathcal{U}^2_{n,\gamma}(x,-t_n)+\mathcal{V}^2_{n,\gamma}(x,-t_n)\right)+\frac{2\left(d_1+d_2\hat{r}_2^2\right)\eta}{k}.
\end{aligned}
$$

That is,

$$
\begin{aligned}
&|U_n(x+\gamma,t)-U_n(x,t)|^2+|V_n(x+\gamma,t)-V_n(x,t)|^2\\
&\le |U_n(x+\gamma,-t_n)-U_n(x,-t_n)|^2+|V_n(x+\gamma,-t_n)-V_n(x,-t_n)|^2+\frac{2\left(d_1+d_2\hat{r}_2^2\right)\eta}{k}.
\end{aligned}
$$

Since $U_n(x,-t_n)$ and $V_n(x,-t_n)$ are uniformly continuous for $x\in\mathbb{R}$, there exists $\delta_3>0$ such that $|U_n(x+\gamma,-t_n)-U_n(x,-t_n)|\le\eta^{1/2}$ and $|V_n(x+\gamma,-t_n)-V_n(x,-t_n)|\le\eta^{1/2}$ whatever $|\gamma|\le\delta_3$. Thus, there exists a positive constant D_5 and for any $\gamma>0$ such that $|\gamma|\le\delta:=\min\{\delta_1,\delta_2,\delta_3\}$, we have that for all $x\in\mathbb{R}$ and $t>-t_n$

$$
\begin{cases}
|U_n(x+\gamma,t)-U_n(x,t)|\le\left(2+\frac{2\left(d_1+d_2\hat{r}_2^2\right)}{k}\right)\eta:=D_5\eta,\\
|V_n(x+\gamma,t)-V_n(x,t)|\le\left(2+\frac{2\left(d_1+d_2\hat{r}_2^2\right)}{k}\right)\eta:=D_5\eta.
\end{cases}
$$

Furthermore, there exist positive constants D_6 and D_7 such that for all $x\in\mathbb{R}$ and $t>-t_n$, it follows that

$$
\begin{aligned}
&\left|\tfrac{\partial U_n}{\partial t}(x+\gamma,t)-\tfrac{\partial U_n}{\partial t}(x,t)\right|\\
\le&\Big|d_1\left(J_1*(U_n(x+\gamma,t)-U_n(x,t))\right)-(d_1-r_1)\left(U_n(x+\gamma,t)-U_n(x,t)\right)\\
&-r_1\left(U_n(x+\gamma,t)+U_n(x,t)\right)\left(U_n(x+\gamma,t)-U_n(x,t)\right)\\
&-\frac{bV_n(x+\gamma,t)}{1+\alpha U_n(x+\gamma,t)}\left(U_n(x+\gamma,t)-U_n(x,t)\right)\\
&-\frac{bU_n(x,t)}{1+\alpha U_n(x+\gamma,t)}\left(V_n(x+\gamma,t)-V_n(x,t)\right)\\
&+\frac{b\alpha U_n(x,t)V_n(x,t)}{[1+\alpha U_n(x+\gamma,t)][1+\alpha U_n(x,t)]}\left(U_n(x+\gamma,t)-U_n(x,t)\right)\Big|\\
\le&\left[\left(2d_1+3r_1+2b\hat{r}_2\right)D_5+bD_5\right]\eta=:D_6\eta,
\end{aligned}
$$

and

$$
\left|\frac{\partial V_n}{\partial t}(x+\gamma,t)-\frac{\partial V_n}{\partial t}(x,t)\right|\le D_7\eta.
$$

By the above priori estimates, the Arzelá-Ascoli theorem and a diagonal extraction process, we can extract a subsequence $t_n\to\infty$ such that

$$
U_n(x,t)\to U_\infty(x,t)\text{ locally uniformly as }n\to\infty,
$$

where $U_\infty(x,t)$ satisfies

$$
\begin{aligned}
&\partial_t U_\infty(x,t)-d_1[(J_1*U_\infty)(x,t)-U_\infty(x,t)]\\
&\quad -r_1U_\infty(x,t)\left[1-U_\infty(x,t)-\frac{b\varepsilon}{r_1(1+\alpha U_\infty(x,t))}\right]\\
&\ge 0,
\end{aligned}
$$

for $(x,t)\in\mathbb{R}\times\mathbb{R}$. Moreover, according to (3.16), we have

$$\liminf_{t_n\to\infty}\inf_{x\in(-\frac{l_2}{2},\frac{l_2}{2}),t'\in[\frac{ct_n}{c'},t_n]} U(x+ct_n+x_3,t')\geq\eta_2. \tag{3.22}$$

Since $U_n(x+x_3,t)=U(x+ct_n+x_3,t+t_n)$, according to (3.22), we obtain that for any $t\leq 0$ with $|t|$ small enough, $\inf_{x\in(-\frac{l_2}{2},\frac{l_2}{2})}U_\infty(x+x_3,t)\geq\eta_2$. Let U_ε be the solution of the following equation

$$\begin{cases}\frac{\partial U_\varepsilon}{\partial t}(x,t)-d_1[(J_1*U_\varepsilon)(x,t)-U_\varepsilon(x,t)]-r_1U_\varepsilon(x,t)\left[1-U_\varepsilon(x,t)-\frac{b\varepsilon}{r_1(1+\alpha U_\varepsilon(x,t))}\right]=0,\\ U_\varepsilon(x,t)=\eta_2 g(x),\end{cases} \tag{3.23}$$

where $g(x)\in C(\mathbb{R},[0,1])$ satisfies $g(x)=1$ for $x\in(-\frac{l_2}{4}+x_3,\frac{l_2}{4}+x_3)$, $g(x)=0$ for $x\in\mathbb{R}\setminus(-\frac{l_2}{2}+x_3,\frac{l_2}{2}+x_3)$, and $g(x)$ is monotone increasing in $x\in(-\frac{l_2}{2}+x_3,\frac{l_2}{4}+x_3)$ and decreasing in $x\in(\frac{l_2}{4}+x_3,\frac{l_2}{2}+x_3)$. Then by comparison principle, we have $U_\infty(x,t)\geq U_\varepsilon(x,t)$. It follows from [24, Theorems 3.1 and 3.2] that $U_\varepsilon(x,t)$ converges locally uniformly to a unique positive equilibrium solution of system (3.23), denoted as p_ε. Hence, $U_\infty(x,\infty)\geq p_\varepsilon$ for $x\in\mathbb{R}$. By the definition of U_∞, we then obtain

$$\liminf_{t\to\infty}U_\infty(x+ct,t)\geq p_\varepsilon \text{ locally uniformly with respect to } x.$$

Since $F(U_\varepsilon,V):=1-U_\varepsilon-\frac{bV}{r_1(1+\alpha U_\varepsilon)}$ decreases monotonically with respect to V and $F(p_\varepsilon,\varepsilon)=0$, we have $p_\varepsilon\leq 1$. Hence, by the arbitrary of ε, it follows that

$$\liminf_{t\to\infty}U(x+ct,t)=1$$

locally uniformly with respect to $x\in\mathbb{R}$. □

Now, we complete the proof of Proposition 3.3. The statement $\lim_{t\to\infty}\sup_{c_1t<x<c_2t}(1-U(x,t))=0$ is a straightforward consequence of Lemma 3.4. Indeed, if this statement is not true, then we can assume that there exist two sequences $\{x_n\}$ and $\{t_n\}$ satisfying $c_1t_n<x_n<c_2t_n$ and $t_n\to\infty$, as $n\to\infty$, such that $\limsup_{n\to+\infty}U(x_n,t_n)<1$. Let $c_n=\frac{x_n}{t_n}$, then $c_n\in(c_1,c_2)\subset(c_V,c_U)$. Thus, there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $\lim_{j\to\infty}c_{n_j}=c$, where c is a real number satisfying $c\in[c_1,c_2]$. By Lemma 3.4, it then follows that $U(x_{n_j},t_{n_j})=U(c_{n_j}t_{n_j},t_{n_j})\to 1$ as $j\to\infty$. This contradicts $\limsup_{n\to+\infty}U(x_n,t_n)<1$ and the proof of the proposition is completed. □

3.3. Spreading of U and V for $0\leq c<c_V$

Proposition 3.5. *Let $d_1>r_1+b\hat{r}_2+\frac{b}{2}+\beta b\hat{r}_2$ and $d_2>\beta b-a+\beta b\hat{r}_2+\frac{b}{2}$. For each $c\in[0,c_V)$, there exists $\varepsilon>0$ such that the solution (U,V) of (1.1) with initial data (u_0,v_0) satisfies:*

$$\liminf_{t\to\infty}\inf_{|x|\leq ct}V(x,t)\geq\varepsilon,$$

$$\limsup_{t\to\infty}\sup_{|x|\leq ct}U(x,t)\leq 1-\varepsilon \text{ and } \liminf_{t\to\infty}\inf_{|x|\leq ct}U(x,t)\geq\varepsilon.$$

We prove this argument in three steps. We first use Lemma 3.6 to prove the "pointwise weak spreading" which illustrates that the U-component of the solution of system (1.1) converges to neither 0 and 1, and the V-component dose not converge to 0. Then Lemma 3.8 shows the "pointwise spreading" which means that the solution is bounded along with the path $x = ct$ by some constant $\varepsilon > 0$ as $t \to +\infty$. In the last step, we use Lemma 3.10 to prove that the spreading is in fact uniform in the interval $[-ct, ct]$ with $0 \le c < c_V$.

Here we assume that c^0 is a fixed constant and satisfies $0 \le c^0 < c_V$.

Lemma 3.6. *(Pointwise weak spreading) There exists $\varepsilon_1(c^0) > 0$ such that for any $(u_0, v_0) \in H$ with $u_0 \not\equiv 0$, for all $c \in [0, c^0]$ and $x \in \mathbb{R}$:*

$$\limsup_{t\to\infty} U(x + ct, t) \ge \varepsilon_1(c^0),$$

and, if moreover $v_0 \not\equiv 0$,

$$\begin{cases} \limsup_{t\to\infty} V(x + ct, t) \ge \varepsilon_1(c^0), \\ \liminf_{t\to\infty} U(x + ct, t) \le 1 - \varepsilon_1(c^0). \end{cases}$$

Remark 3.7. It follows from this statement that $\varepsilon_1(c^0)$ can be chosen to be nonincreasing with respect to c^0.

Proof of Lemma 3.6. Note that when $v_0 \equiv 0$, the equation for U becomes a standard monostable type equation with nonlocal dispersal. By Proposition 2.5, $U(x + ct, t)$ converges to 1 as $t \to \infty$ for any $x \in \mathbb{R}$ and $c \in [0, c_U)$. Therefore, we only need to consider the case when v_0 is not trivial.

We argue by contradiction by assuming that there exist sequences

$$\{(u_{0,n}, v_{0,n})\}_{n\ge 0} \in H, \{c_n\}_{n\ge 0} \subset [0, c^0] \text{ and } \{x_n\}_{n\ge 0} \subset \mathbb{R},$$
$$\{t_n\}_{n\ge 0} \subset [0, \infty) \text{ such that } t_n \to +\infty,$$

such that $u_{0,n}, v_{0,n} \not\equiv 0$ and one of the following three options holds true:

$$\forall t \ge t_n,\ U_n(x_n + c_n t, t) \le \frac{1}{n}, \tag{3.24}$$

$$\forall t \ge t_n,\ U_n(x_n + c_n t, t) \ge 1 - \frac{1}{n}, \tag{3.25}$$

or

$$\forall t \ge t_n,\ V_n(x_n + c_n t, t) \le \frac{1}{n}, \tag{3.26}$$

where (U_n, V_n) is the solution of system (1.1) with initial value $(u_{0,n}, v_{0,n})$. Without loss of generality, we assume that $c_n \to c_\infty \in [0, c^0]$.

We first prove that (3.24) implies (3.26). Choose any sequence $s_n \ge t_n$. By the arguments similar to the Lemma 3.4, we can extract a subsequence such that the following convergence holds locally uniform in $(x, t) \in \mathbb{R} \times \mathbb{R}$

$$\begin{cases} \lim\limits_{n\to\infty} U_n\left(x_n + c_n\left(s_n + t\right) + x, s_n + t\right) = U_\infty(x,t), \\ \lim\limits_{n\to\infty} V_n\left(x_n + c_n\left(s_n + t\right) + x, s_n + t\right) = V_\infty(x,t), \end{cases}$$

where the limit function (U_∞, V_∞) is an entire solution of the following system, which is the same as (1.1) but there are some additional drift terms:

$$\begin{cases} \partial_t U_\infty = d_1(J_1 * U_\infty - U_\infty) + c_\infty \nabla U_\infty + U_\infty\left[r_1(1-U_\infty) - \frac{bV_\infty}{1+\alpha U_\infty}\right], \\ \partial_t V_\infty = d_2(J_2 * V_\infty - V_\infty) + c_\infty \nabla V_\infty + V_\infty\left[\frac{\beta b U_\infty}{1+\alpha U_\infty} - aV_\infty - a\right]. \end{cases} \tag{3.27}$$

It is easy to see that $U_\infty \geq 0$ and we deduce from (3.24) that $U_\infty(0,0) = 0$. According to the comparison principle in Theorem 2.1, we obtain that $U_\infty \equiv 0$, hence V_∞ satisfies

$$\partial_t V_\infty = d_2(J_2 * V_\infty - V_\infty) + c_\infty \nabla V_\infty - aV_\infty - aV_\infty^2.$$

It is easy to verify that for any $t_0 \in \mathbb{R}$, the function $(x,t) \mapsto \hat{r}_2 e^{-a(t+t_0)}$ is an upper solution of the above equation, for any $t > -t_0$. Due to $V_\infty(x, -t_0) \leq \hat{r}_2$ for any $t_0 \in \mathbb{R}^+$, we deduce that $V_\infty(x,0) \leq \hat{r}_2 e^{-at_0}$. Then we have $V_\infty(x,0) \equiv 0$ as $t_0 \to \infty$. Thus,

$$V_n(x_n + c_n s_n, s_n) \to 0 \text{ as } n \to \infty.$$

And since the choice of the sequence s_n is arbitrary, (3.26) holds true.

Now, we give the following claim.

Claim* *In both cases, that is either (3.25) or (3.26) holds, then there exists a sequence $\{t_n'\}_{n\geq 0}$ such that $t_n' \geq t_n$ and for any $R > 0$,*

$$\lim_{n\to\infty} \sup_{t\geq 0, x\in[-R,R]} |1 - U_n(x_n + c_n(t_n' + t) + x, t_n' + t)| = 0. \tag{3.28}$$

We will prove the Claim* in next subsection 3.4. We continue the proof of Lemma 3.6 and derive the contradiction mainly through (3.28). Without loss of generality, we first assume that $t_n' = t_n$ for all $n \in \mathbb{N}$. We can see that for any $R > 0$ and $\delta > 0$, then for any n large enough, any $t \geq 0$ and $x \in \mathbb{R}$:

$$U_n(x_n + c_n(t_n + t) + x, t_n + t) \geq (1-\delta)\chi_{[-R,R]}(x) =: \underline{U}(x).$$

Then applying the comparison principle, we have that

$$V_n(x_n + c_n(t_n + t) + x, t_n + t) \geq \underline{V}^n(x,t), \quad \forall t \geq 0, \quad x \in \mathbb{R},$$

where $\underline{V}^n$ is the solution of

$$\begin{cases} \partial_t \underline{V}^n(x,t) = d_2(J_2 * \underline{V}^n - \underline{V}^n)(x,t) + c_n \nabla \underline{V}^n(x,t) + \underline{V}^n(x,t)\left[\frac{\beta b \underline{U}(x)}{1+\alpha \underline{U}(x)} - a - a\underline{V}^n(x,t)\right], \\ \underline{V}^n(x,0) = V_n\left(x_n + c_n t_n + x, t_n\right). \end{cases} \tag{3.29}$$

Similar to the proof of Claim* (see Section 3.4 for details), we obtain that the function $\phi_{R,\alpha_2,\eta}(x) = \eta e^{-\alpha_2 x}\cos\left(\frac{\pi x}{2R}\right)$ is a sub-solution of (3.29), provided that R is large, $\alpha_2 \in (0,\alpha^*)$ and $\eta \in (0,\eta_0]$, using the fact that $c^0 < c_V$. It follows from the comparison principle that

$$V_n(x_n + c_n(t_n+t)+x, t_n+t) \geq \underline{V}^n(x,t) \geq \Phi(x,t), \quad \forall t \geq 0, \quad x \in \mathbb{R}, \tag{3.30}$$

wherein $\Phi = \Phi(x,t)$ is the solution of (3.29) with initial data $\phi_{R,\alpha_2,\eta}$. Since function $\Phi = \Phi(x,t)$ is increasing in time and bounded by $\hat{r}_2$, it converges to some positive stationary solution that does not depend on α_2, denoted by $q_{n,R,\delta}$. Furthermore, using a prior estimate and Arzelá-Ascoli theorem, we can extract a subsequence of $q_{n,R,\delta}$ converging as $n \to \infty$, $R \to \infty$ and $\delta \to 0$ to a stationary state q_∞ of

$$d_2(J_2 * q_\infty - q_\infty) + c_\infty \nabla q_\infty + q_\infty\left[\frac{\beta b}{1+\alpha} - a - aq_\infty\right] = 0. \tag{3.31}$$

Due to $\cos\left(\frac{\pi x}{2R}\right) \to 1$ locally uniformly as $R \to \infty$, q_∞ is bounded and positive. We claim that

$$\inf_{x\in\mathbb{R}} q_\infty(x) > 0.$$

Indeed, notice that if $q_\infty \equiv q_\infty(x)$ is a stationary solution of (3.31), then the map $V(x,t) = q_\infty(x - c_\infty t)$ satisfies

$$\partial_t V(x,t) - d_2(J_2 * V - V)(x,t) = V(x,t)\left[\frac{\beta b}{1+\alpha} - a - aV(x,t)\right]. \tag{3.32}$$

It follows from Proposition 2.5 that

$$\liminf_{t\to\infty} \inf_{|x|<c_\infty t} V(x,t) = \hat{r}_2 > 0, \quad c_\infty \in [0, c_V).$$

Namely, for each $x \in \mathbb{R}$ and $c_\infty \in [0, c_V)$, we have

$$\liminf_{t\to\infty} V(x + c_\infty t, t) > 0.$$

Note that $V(x + c_\infty t, t) = q_\infty(x)$ for any $x \in \mathbb{R}$ and $t \geq 0$, then we complete the claim. Using (3.30), we have

$$\liminf_{t\to+\infty} V_n\left(x_n + c_n\left(t_n+t\right)+x, t_n+t\right) \geq \frac{\inf q_\infty}{2} > 0, \quad x \in [-R, R],$$

where $R > 0$ can be chosen large enough for n large enough. But thanks to the above lower estimate, we obtain that

$$\limsup_{t\to+\infty} U_n\left(x_n + c_n\left(t_n+t\right)+x, t_n+t\right) \leq \lim_{t\to+\infty} \overline{U}_n(x,t),$$

where $\overline{U}_n$ denotes the solution of

$$\begin{cases} \partial_t \overline{U}_n = d_1(J_1 * \overline{U}_n - \overline{U}_n) + c_n \nabla \overline{U}_n + \overline{U}_n[r_1(1-\overline{U}_n) - \frac{\inf q_\infty}{2(1+\alpha \overline{U}_n)} b\chi_{[-R,R]}(x)], \\ \overline{U}_n(x,0) = 1. \end{cases}$$

Since 1 is a super-solution, the limit of $\overline{U}_n$ as $t \to +\infty$ is well-defined and decreases with respect to time. Moreover, we claim that this limit stays locally away from 1, uniformly as $n \to \infty$. Indeed, we can extract a subsequence of $\overline{U}_n(x,t)$ that converges locally uniformly to the solution $\overline{U}_\infty(x,t)$ of the same problem where c_n is replaced by c_∞. Since 1 still is a strict super-solution, we have that $\overline{U}_\infty(0,1) < 1$. Therefore

$$\limsup_{n\to\infty} \overline{U}_n(0,1) < 1,$$

and

$$\limsup_{n\to\infty} \lim_{t\to+\infty} \overline{U}_n(0,t) < 1,$$

since $\overline{U}_n$ decreases with respect to time. This contradicts Claim* and the proof of Lemma 3.6 is completed. □

Lemma 3.8. *(Pointwise spreading) There exists* $\varepsilon_2(c^0) > 0$ *such that, for any* $(u_0, v_0) \in H$ *with* $u_0 \not\equiv 0$ *and* $v_0 \not\equiv 0$, *for all* $c \in [0, c^0]$ *and any* $x \in \mathbb{R}$*:*

$$\begin{cases} \liminf_{t\to+\infty} V(x+ct,t) \geq \varepsilon_2\left(c^0\right), \\ \liminf_{t\to+\infty} U(x+ct,t) \geq \varepsilon_2\left(c^0\right), \\ \limsup_{t\to+\infty} U(x+ct,t) \leq 1 - \varepsilon_2\left(c^0\right). \end{cases}$$

Remark 3.9. Similar to Remark *3.7*, we have that $\varepsilon_2(c^0)$ can be chosen to be nonincreasing with respect to c^0. We mainly use the idea of uniform persistence theory in dynamical systems to prove this lemma, see [14,50,51].

Proof of Lemma 3.8. We argue by contradiction to prove the first assertion, i.e. V spreads away from 0. We assume that there are sequences $(u_{0,n}, v_{0,n}) \in H$ with $u_{0,n} \not\equiv 0$ and $v_{0,n} \not\equiv 0$, $c_n \in [0, c^0]$ and $x_n \in \mathbb{R}$, such that

$$\liminf_{t\to+\infty} V_n(x_n + c_n t, t) < \frac{1}{n}.$$

Without loss of generality, we assume that $c_\infty = \lim_{n\to\infty} c_n \in [0, c^0]$. By Lemma 3.6, there exist two sequences $t_n \to \infty$ and $s_n \in \mathbb{R}^+$ such that for each $n \geq 0$

$$\begin{cases} V_n(x_n + c_n t_n, t_n) = \frac{\varepsilon}{2}, \\ V_n(x_n + c_n t, t) \leq \frac{\varepsilon}{2}, \quad \forall t \in [t_n, t_n + s_n], \\ V_n(x_n + c_n(t_n + s_n), t_n + s_n) = \frac{1}{n}, \end{cases}$$

where $\varepsilon = \varepsilon_1(c^0)$ is provided by Lemma 3.6.

Similar to the previous proof, we extract a converging subsequence

$$\begin{cases} \lim_{n\to\infty} U_n(x_n + c_n(t_n + s_n) + x, t_n + s_n + t) = U_\infty(x,t), \\ \lim_{n\to\infty} V_n(x_n + c_n(t_n + s_n) + x, t_n + s_n + t) = V_\infty(x,t). \end{cases}$$

The above convergence is locally uniform in $(x,t) \in \mathbb{R} \times \mathbb{R}$ and the limit function (U_∞, V_∞) is an entire solution of (1.1). We choose t_n and s_n such that $V_\infty(0,0) = 0$. Using the comparison principle, we obtain that $V_\infty \equiv 0$. In particular, the sequence s_n is unbounded. Indeed, assume by contraction that $\lim_{n\to\infty} s_n = s < +\infty$, then we have

$$\lim_{n\to+\infty} V_n(x_n + c_n(t_n + t), t_n + t) = 0 \quad \forall t \in [0, s_n],$$

which is impossible because

$$\lim_{n\to+\infty} V_n(x_n + c_n t_n, t_n) = \frac{\varepsilon}{2} > 0.$$

Thus we assume that $s_n \to +\infty$ as $n \to +\infty$.

We now consider the limit functions

$$\begin{cases} \tilde{U}(x,t) = \lim_{n\to+\infty} U_n(x_n + c_n t_n + x, t_n + t), \\ \tilde{V}(x,t) = \lim_{n\to+\infty} V_n(x_n + c_n t_n + x, t_n + t), \end{cases}$$

which are well defined as a result of global boundedness and a prior estimate. The pair $(\tilde{U}, \tilde{V})$ is a global in time solution of system (1.1), and $\tilde{V}(0,0) = \frac{\varepsilon}{2} > 0$.

We claim that $\tilde{U}(x,0) \not\equiv 0$. Indeed, we argue by contradiction by assuming that $\tilde{U}(x,0) \equiv 0$, thus $\tilde{U} \equiv 0$ thanks to the comparison principle. Then $\tilde{V}$ satisfies

$$\partial_t \tilde{V}(x,t) = d_2(J_2 * \tilde{V} - \tilde{V})(x,t) - a\tilde{V}(x,t) - a\tilde{V}^2(x,t). \tag{3.33}$$

For any $t_0 \in \mathbb{R}$, define $\hat{V}(x,t) = \hat{r}_2 \mathrm{e}^{-a(t+t_0)}$. It is easy to check that $\hat{V}(x,t)$ is an upper solution of (3.33), for any $t > -t_0$. Since $\tilde{V}(x,-t_0) \le \hat{r}_2$ for any $t_0 \in \mathbb{R}^+$, we have $\tilde{V}(x,0) \le \hat{r}_2 \mathrm{e}^{-at_0}$. Then we obtain that $\tilde{V}(x,0) \equiv 0$ as $t_0 \to \infty$. By the comparison principle, we get that $\tilde{V} \equiv 0$ which contradicts $\tilde{V}(0,0) > 0$.

Next we take $(\tilde{U}, \tilde{V})$ as a solution of system (1.1) with initial data

$$\left(\tilde{U}_0, \tilde{V}_0\right) := \lim_{n\to+\infty} (U_n(x_n + c_n t_n + x, t_n), V_n(x_n + c_n t_n + x, t_n)) \in H,$$

and $\tilde{U}_0 \not\equiv 0$, as well as $\tilde{V}_0 \not\equiv 0$. By Lemma 3.6, we obtain that

$$\forall x \in \mathbb{R}, \ \limsup_{t\to+\infty} \tilde{V}(x + ct, t) \ge \varepsilon, \ \text{for any } c \in [0, c^0].$$

On the other hand, for all $t \in [0, s_n)$,

$$V_n(x_n + c_n t_n + c_n t, t_n + t) \le \frac{\varepsilon}{2}.$$

Due to $s_n \to +\infty$, it follows from the locally uniform convergence that

$$\tilde{V}(c_\infty t, t) \le \frac{\varepsilon}{2}, \forall t \ge 0,$$

which contradicts the inequality above provided by Lemma 3.6.

The second and third assertions, namely the spreading properties of U, can be proved with similar arguments. □

Lemma 3.10. *(Uniform spreading) Let the initial data* $(u_0, v_0) \in H$ *with* $u_0 \not\equiv 0$ *and* $v_0 \not\equiv 0$ *be given. Then for any* $0 \le c^0 < \min\{c_U, c_V\}$*, there exists* $\varepsilon > 0$ *such that*

$$\begin{cases} \liminf\limits_{t\to+\infty} \inf\limits_{|x|\le c^0 t} V(x,t) \ge \varepsilon, \\ \liminf\limits_{t\to+\infty} \inf\limits_{|x|\le c^0 t} U(x,t) \ge \varepsilon, \\ \limsup\limits_{t\to+\infty} \sup\limits_{|x|\le c^0 t} U(x,t) \le 1-\varepsilon. \end{cases}$$

Proof. We argue by contradiction by assuming that there exist $t_n \to +\infty$, $c_n \in [0, c^0]$ such that

$$V(c_n t_n, t_n) \to 0. \tag{3.34}$$

Without loss of generality, we assume that $c_n \to c_\infty \in [0, c^0]$ as $n \to +\infty$. Choose some small $\delta > 0$ such that $c_\infty + \delta < \min\{c_U, c_V\}$, and define the sequence

$$t_n' := \frac{c_n t_n}{c_\infty + \delta} \in [0, t_n), \quad \forall n \ge 0.$$

First, we consider the case when the sequence $\{c_n t_n\}_{n\ge 0}$ is bounded, which may happen if $c_\infty = 0$. Then we can extract a subsequence such that $c_n t_n \to x_\infty \in \mathbb{R}$ as $n \to +\infty$. We consider the functions

$$\hat{U}_n(x,t) = U(x + c_n t_n, t_n + t), \quad \hat{V}_n(x,t) = V(x + c_n t_n, t_n + t), \tag{3.35}$$

and

$$\lim_{n\to\infty} \hat{U}_n(x,t) := \hat{U}(x,t), \quad \lim_{n\to\infty} \hat{V}_n(x,t) := \hat{V}(x,t). \tag{3.36}$$

Using (3.34), (3.35) and (3.36), we have that

$$\hat{V}(0,0) = \lim_{n\to\infty} V(c_n t_n, t_n) = 0.$$

Then by the comparison principle, we can extract a subsequence such that

$$V(c_n t_n + x, t_n + t) \to 0, \text{ as } n \to +\infty,$$

locally uniformly. Moreover, we have that $V(0, t_n) \to 0$ as $n \to +\infty$, which contradicts Lemma 3.8 with $c = 0$.

On the other hand, we assume that $t_n' \to +\infty$. Then applying Lemma 3.8, one has that

$$V((c_\infty + \delta)t_n', t_n') \geq \varepsilon, \tag{3.37}$$

for each n large enough, where $\varepsilon = \varepsilon_2(c_\infty + \delta)$ is the constant provided by Lemma 3.8.

We consider the functions

$$\tilde{U}_n(x,t) = U(x + c_n t_n, t_n' + t), \quad \tilde{V}_n(x,t) = V(x + c_n t_n, t_n' + t), \tag{3.38}$$

and

$$\lim_{n\to\infty} \tilde{U}_n(x,t) := \tilde{U}(x,t), \quad \lim_{n\to\infty} \tilde{V}_n(x,t) := \tilde{V}(x,t).$$

From (3.38), we can rewrite (3.34) and (3.37) as

$$\tilde{V}_n(0,0) \geq \varepsilon, \quad \tilde{V}_n(0, t_n - t_n') \to 0.$$

Now we give the following sequences

$$\begin{aligned} &\tilde{t}_n := \sup\left\{0 \leq t \leq t_n - t_n' | \tilde{V}_n(0,t) \geq \tfrac{\varepsilon}{2}\right\} \in \left(0, t_n - t_n'\right), \\ &\tilde{s}_n := t_n - t_n' - \tilde{t}_n. \end{aligned}$$

Then the following properties can be derived,

$$\begin{aligned} &\tilde{V}_n\left(0, \tilde{t}_n\right) = \tfrac{\varepsilon}{2}, \\ &\tilde{V}_n(0,t) \leq \tfrac{\varepsilon}{2}, \quad \forall t \in \left[\tilde{t}_n, \tilde{t}_n + \tilde{s}_n\right], \\ &\tilde{V}_n\left(0, \tilde{t}_n + \tilde{s}_n\right) \to 0 \text{ as } n \to +\infty. \end{aligned}$$

Similar to the proof of Lemma 3.8, we can draw a contradiction with Lemma 3.6. Thus we obtain that

$$\liminf_{t\to+\infty} \inf_{|x|\leq c^0 t} V(x,t) \geq \varepsilon.$$

The second and third statements, namely

$$\liminf_{t\to+\infty} \inf_{|x|\leq c^0 t} U(x,t) \geq \varepsilon,$$

and

$$\limsup_{t\to+\infty} \sup_{|x|\leq c^0 t} U(x,t) \leq 1 - \varepsilon,$$

can be proved with similar arguments. Therefore, we complete the proof of Lemma 3.10. □

3.4. Proof of Claim*

In this subsection, we give the proof of Claim* introduced in subsection 3.3.

Proof of Claim*. First, we consider the case when (3.25) holds true. We will prove that Claim* holds true for the sequence $t'_n = t_n$. We argue by contradiction by assuming that for some $R > 0$, there is $\delta > 0$, for any sequences $s_n \geq t_n$ and $x'_n \in [-R, R]$ such that

$$U_n(x_n + c_n s_n + x'_n, s_n) \leq 1 - \delta. \tag{3.39}$$

Using Arzelá-Ascoli theorem again, we extract a converging subsequence

$$\begin{cases} \lim\limits_{n\to\infty} U_n\left(x_n + c_n\left(s_n + t\right) + x, s_n + t\right) = U_\infty(x, t), \\ \lim\limits_{n\to\infty} V_n\left(x_n + c_n\left(s_n + t\right) + x, s_n + t\right) = V_\infty(x, t), \end{cases}$$

where the limit function (U_∞, V_∞) is an entire solution of (3.27). It is easy to see that $U_\infty \leq 1$ and by (3.25), we have that $U_\infty(0, 0) = 1$. Therefore, according to the comparison principle in Theorem 2.1, we obtain that $U_\infty \equiv 1$ (as well as $V_\infty \equiv 0$). But, since the sequence $\{x'_n\} \subset [-R, R]$ is relatively compact, it follows from (3.39) that

$$U_\infty(x'_\infty, 0) \leq 1 - \delta,$$

where x'_∞ is an accumulation point of the sequence $\{x'_n\}$. This contradicts $U_\infty \equiv 1$, thus Claim* holds true under condition (3.25).

Now we consider the case when (3.26) holds true. We first verify that for any $R > 0$, the following formula holds true

$$\lim_{n\to\infty} V_n\left(x_n + c_n\left(t_n + t\right) + x, t_n + t\right) = 0 \text{ uniformly on } [-R, R] \times [0, \infty). \tag{3.40}$$

Indeed, if this is not true, then for some $R > 0$, there are $\delta > 0$, $s_n \geq t_n$ and $x'_n \in [-R, R]$ such that

$$V_n(x_n + c_n s_n + x'_n, s_n) \geq \delta.$$

Using again Arzelá-Ascoli theorem, we extract a subsequence, that is, as $n \to \infty$:

$$\begin{cases} U_n\left(x_n + c_n\left(s_n + t\right) + x, s_n + t\right) \to U_\infty(x, t), \\ V_n\left(x_n + c_n\left(s_n + t\right) + x, s_n + t\right) \to V_\infty(x, t), \end{cases}$$

where the above convergence holds locally uniformly and (U_∞, V_∞) is an entire solution of (3.27). By the comparison principle in Theorem 2.1 and (3.26), we obtain that $V_\infty \equiv 0$, which contradicts $V_\infty(x'_\infty, 0) > \delta$, where x'_∞ is an accumulation point of the sequence $\{x'_n\}$. Thus, (3.40) holds true.

Now we are ready to show (3.28). We have just shown that for any $R > 0$ and $\delta > 0$, then for any sufficiently large n it follows that for each $x \in \mathbb{R}$ and $t \geq 0$:

$$V_n\left(x_n+c_n\left(t_n+t\right)+x,t_n+t\right)\le \hat{r}_2\chi_{\mathbb{R}\setminus[-R,R]}(x)+\delta\chi_{[-R,R]}(x)=:\overline{V}(x). \tag{3.41}$$

Applying the comparison principle in Theorem 2.1 and (3.41), we deduce that for each $R>0$, $\delta>0$ and n large enough

$$U_n\left(x_n+c_n\left(t_n+t\right)+x,t_n+t\right)\ge \underline{U}^n(x,t),\quad \forall t\ge 0,\quad x\in\mathbb{R}, \tag{3.42}$$

where $\underline{U}^n$ is the solution of

$$\begin{cases}\partial_t\underline{U}^n(x,t)=d_1(J_1*\underline{U}^n-\underline{U}^n)(x,t)+c_n\nabla\underline{U}^n(x,t)+\underline{U}^n\left[r_1(1-\underline{U}^n(x,t))-\frac{b\overline{V}(x)}{1+\alpha\underline{U}^n(x,t)}\right],\\ \underline{U}^n(x,0)=U_n\left(x_n+c_nt_n+x,t_n\right).\end{cases} \tag{3.43}$$

Consider the following operators for $x\in[-R,R]$ and $R>0$

$$\begin{aligned}Q_{c_n}[W](x,t):=d_1\left[\int_{\mathbb{R}}J_1(x-y)W(y,t)\mathrm{d}y-W(x,t)\right]+c_n\nabla W(x,t)\\ +W(x,t)\left[r_1(1-W(x,t))-\frac{b\delta}{1+\alpha W(x,t)}\right],\end{aligned}$$

and

$$L_{c_n}[W](x,t):=d_1\int_{\mathbb{R}}J_1(x-y)W(y,t)\mathrm{d}y+c_n\nabla W(x,t)+m_1W(x,t), \tag{3.44}$$

where m_1 will be determined later. Let

$$\phi_{R,\alpha_2,\eta}(x)=\begin{cases}\eta\mathrm{e}^{-\alpha_2x}\cos\left(\frac{\pi x}{2R}\right), & x\in(-R,R),\alpha_2>0,0<\eta\le\eta_0,\\ 0, & x\in\mathbb{R}\setminus(-R,R),\end{cases} \tag{3.45}$$

where $\eta_0\in(0,+\infty)$ and we assume that α_2 and η are two independent constants. We claim that for R large enough

$$J_1*\phi_{R,\alpha_2,\eta}(x)=\eta\int_{-R}^{R}J_1(x-y)\mathrm{e}^{-\alpha_2y}\cos\left(\frac{\pi y}{2R}\right)\mathrm{d}y\ge\eta\int_{-\infty}^{\infty}J_1(x-y)\mathrm{e}^{-\alpha_2y}\cos\left(\frac{\pi y}{2R}\right)\mathrm{d}y. \tag{3.46}$$

Indeed, due to anyhow $J_1*\phi_{R,\alpha_2,\eta}(x)\ge0$, we take without loss of generality that $x\in(-R,R)$. In order to make (3.46) valid we have to show that for $y\in\mathbb{R}\setminus(-R,R)$ either $\cos\left(\frac{\pi y}{2R}\right)\le0$ or $J_1(x-y)=0$. We assume J_1 has a compact support, then there exists K such that supp $J_1\subset[-K,K]$. If $x\in(-R,R)$ and $|x-y|\le K$, then $y\in(-R-K,R+K)\subset(-3R,3R)$ when $K\le2R$. Thus, we obtain that $\cos\left(\frac{\pi y}{2R}\right)\le0$ for $y\in[-3R,-R]\cup[R,3R]$. Moreover, since supp $J_1\subset[-K,K]\subset(-2R,2R)$ when $K<2R$, we obtain that $J_1(x-y)=0$ for $y\in\mathbb{R}\setminus(-3R,3R)$. Thus, we deduce that

$$\eta\int_{-\infty}^{\infty}J_1(x-y)\mathrm{e}^{-\alpha_2y}\cos\left(\frac{\pi y}{2R}\right)\mathrm{d}y$$

$$=\left(\int_{-\infty}^{-3R}+\int_{-3R}^{-R}+\int_{-R}^{R}+\int_{R}^{3R}+\int_{3R}^{\infty}\right) J_1(x-y)\eta \mathrm{e}^{-\alpha_2 y}\cos\left(\frac{\pi y}{2R}\right)\mathrm{d}y$$

$$\leq \eta\int_{-R}^{R} J_1(x-y)\mathrm{e}^{-\alpha_2 y}\cos\left(\frac{\pi y}{2R}\right)\mathrm{d}y.$$

Taking (3.45) into (3.44) and using (3.46), we obtain that

$$\begin{aligned}
&L_{c_n}[\phi_{R,\alpha_2,\eta}](x)\\
\geq& c_n\left[-\alpha_2\phi_{R,\alpha_2,\eta}-\frac{\pi\eta}{2R}\mathrm{e}^{-\alpha_2 x}\sin\left(\frac{\pi x}{2R}\right)\right]+m_1\phi_{R,\alpha_2,\eta}+d_1\eta\int_{-\infty}^{\infty} J_1(x-y)\mathrm{e}^{-\alpha_2 y}\cos\left(\frac{\pi y}{2R}\right)\mathrm{d}y\\
=&\left[-c_n\alpha_2+m_1+d_1\int_{\mathbb{R}}\mathrm{e}^{\alpha_2 y}J_1(y)\cos\left(\frac{\pi y}{2R}\right)\mathrm{d}y\right]\phi_{R,\alpha_2,\eta}\\
&+\left[-\frac{\pi}{2R}c_n+d_1\int_{\mathbb{R}}\mathrm{e}^{\alpha_2 y}J_1(y)\sin\left(\frac{\pi y}{2R}\right)\mathrm{d}y\right]\eta\mathrm{e}^{-\alpha_2 x}\sin\left(\frac{\pi x}{2R}\right).
\end{aligned}$$

Therefore, $L_{c_n}\left[\phi_{R,\alpha_2,\eta}\right]>0$ on $x\in[-R,R]$ if the following two conditions are satisfied:

$$c_n<\frac{1}{\alpha_2}\left[m_1+d_1\int_{\mathbb{R}}\mathrm{e}^{\alpha_2 y}J_1(y)\cos\left(\frac{\pi y}{2R}\right)\mathrm{d}y\right]=:\mathcal{A}_{m_1}(\alpha_2,R), \tag{3.47}$$

$$c_n=\frac{2Rd_1}{\pi}\left[\int_{\mathbb{R}}\mathrm{e}^{\alpha_2 y}J_1(y)\sin\left(\frac{\pi y}{2R}\right)\mathrm{d}y\right]=:\mathcal{B}(\alpha_2,R). \tag{3.48}$$

We first establish some properties of the functions $\mathcal{A}_{m_1}$ and $\mathcal{B}(\alpha_2,R)$. As $R\to\infty$, we have locally uniform convergence of

$$\mathcal{A}_{m_1}(\alpha_2,R)\to A_{m_1}(\alpha_2)=\frac{m_1+d_1\int_{\mathbb{R}}\mathrm{e}^{\alpha_2 y}J_1(y)\mathrm{d}y}{\alpha_2},$$

$$\mathcal{B}(\alpha_2,R)\to B(\alpha_2):=d_1\int_{\mathbb{R}} y\mathrm{e}^{\alpha_2 y}J_1(y)\mathrm{d}y.$$

Differentiation gives

$$A'_{m_1}(\alpha_2)=\left(B(\alpha_2)-A_{m_1}(\alpha_2)\right)/\alpha_2, \tag{3.49}$$

$$B'(\alpha_2)=d_1\int_{\mathbb{R}} J(y)\mathrm{e}^{\alpha_2 y}y^2\mathrm{d}y>0.$$

It follows from the properties of the function $A_{m_1}(\alpha_2)$ that it achieves infimum. Then, there exists $\alpha^* > 0$ such that $A_{m_1}(\alpha^*) = \inf_{\alpha_2>0} A_{m_1}(\alpha_2)$. By the definition of α^* and (3.49), we obtain that $B(\alpha^*) = A_{m_1}(\alpha^*)$. Since B is an increasing function, thus $B(\alpha_2) < B(\alpha^*)$ for $0 < \alpha_2 < \alpha^*$. Then we have

$$A_{m_1}(\alpha_2) > A_{m_1}(\alpha^*) = B(\alpha^*) > B(\alpha_2) \text{ for } 0 < \alpha_2 < \alpha^*. \tag{3.50}$$

In addition, we define $c^* := A_{m^*}(\alpha^*)$ with $m^* = r_1 - d_1 - b\delta - r_1\varepsilon$ and $r_1 - b\delta - r_1\varepsilon > 0$ for small enough constant $\varepsilon > 0$. We can choose $m_1 < m^*$ such that $0 \le c_n \le c^0 < A_{m_1}(\alpha^*) < A_{m^*}(\alpha^*) = c^*$ for any $n \ge 0$ and $c^0 < c_V$, thus we have $0 \le c_n \le c^0 < \min\{c^*, c_V\}$. Note that $B(0) < c^*$ and $B(0) = 0$, then we have $c_n > B(0)$. Therefore, combined with (3.50), we can choose $\tilde{c}_1, \tilde{c}_2, \tilde{\delta}, R > 0$ such that

$$B(\tilde{c}_1) + \tilde{\delta} < c_n < B(\tilde{c}_2) - \tilde{\delta} \text{ and } |\mathcal{B}(\alpha_2, R) - B(\alpha_2)| < \tilde{\delta}.$$

It follows from the continuity of $\mathcal{B}(\alpha_2, R)$ and $B(\alpha_2)$ that there exists some $\alpha_2(R)$ such that $\mathcal{B}(\alpha_2(R), R) = c_n$ for all large enough R. Obviously, we can choose R large enough such that $\mathcal{A}_{m_1}(\alpha_2(R), R) > c_n$. Thus, we prove that (3.48) and (3.47) hold true.

Hence we obtain $L_{c_n}[\phi_{R,\alpha_2,\eta}](x) > 0$ for $x \in [-R, R]$. Note that $r_1 W(1-W) - \frac{b\delta W}{1+\alpha W} \ge (r_1 - r_1\varepsilon - b\delta)W$ for $0 \le W \le \varepsilon$ and $m_1 < r_1 - d_1 - b\delta - r_1\varepsilon$. Therefore, we have $Q_{c_n}\left[\phi_{R,\alpha_2,\eta}\right] > L_{c_n}\left[\phi_{R,\alpha_2,\eta}\right] > 0$ on $x \in [-R, R]$, namely,

$$\begin{aligned}&-d_1(J_1 * \phi_{R,\alpha_2,\eta} - \phi_{R,\alpha_2,\eta}) - c_n \nabla\phi_{R,\alpha_2,\eta} - \phi_{R,\alpha_2,\eta}\left(r_1 - r_1\phi_{R,\alpha_2,\eta} - \frac{b\delta}{1+\alpha\phi_{R,\alpha_2,\eta}}\right)\\ &< 0 \text{ in } [-R, R].\end{aligned}$$

Furthermore since supp $\phi_{R,\alpha_2,\eta} \subset [-R, R]$, $\phi_{R,\alpha_2,\eta}$ is also a sub-solution of equation (3.43) satisfied by $\underline{U}^n$, the solution $\Phi_{R,\alpha_2,\eta}(x,t)$ of (3.43) with the initial data $\Phi_{R,\alpha_2,\eta}(\cdot, 0) = \phi_{R,\alpha_2,\eta}(\cdot)$ is increasing in time and converges to some positive stationary solution that we denote by $p_{n,R,\delta}$. Now we will show that it does not depend on the choice of $\alpha_2 \in (0, \alpha^*)$ and $\eta \in (0, \eta_0]$. To do this, we simplify the notation of the stationary solution to p_{α_2}, while n, R, δ and η are fixed for the time because α_2 and η are independent. By the comparison principle, we know that $p_{\alpha_2} \le p_{\alpha'}$ for any $\alpha_2, \alpha' \in (0, \alpha^*)$ such that $\alpha_2 \ge \alpha'$. Next we argue by contradiction by assuming that there is $0 < \alpha_3 < \alpha_1 < \alpha^*$ with $p_{\alpha_1} \not\equiv p_{\alpha_3}$. Hence by the comparison principle in Theorem 2.1, we deduce that $p_{\alpha_1} < p_{\alpha_3}$. In addition, there exists a point $x_0 \in (-R, R)$ such that

$$\Phi_{R,\alpha_1,\eta}(x_0, 0) < p_{\alpha_3}(x_0).$$

Indeed, if it is not true then $\Phi_{R,\alpha_1,\eta}(x, 0) \ge p_{\alpha_3}(x)$ for all $x \in \mathbb{R}$, which yields $p_{\alpha_1}(x) \ge p_{\alpha_3}(x)$, a contradiction. We now consider

$$\hat{\alpha} = \inf\{\alpha_2 \le \alpha_3 : \Phi_{R,\alpha_2,\eta}(x, 0) \le p_{\alpha_3}(x), \forall x \in \mathbb{R}\}.$$

Then by the comparison principle in Theorem 2.1, we have that

$$\Phi_{R,\hat{\alpha},\eta}(x, 0) < \Phi_{R,\hat{\alpha},\eta}(x, t) < p_{\alpha_3}(x), \forall t > 0, \forall x \in \mathbb{R}.$$

On the other hand, by the definition of $\hat{\alpha}$ and the function $\Phi_{R,\hat{\alpha},\eta}$ having compact support $[-R, R]$, there is $x_0 \in [-R, R]$ such that $\Phi_{R,\hat{\alpha},\eta}(x_0, 0) = p_{\alpha_3}(x_0)$, a contradiction. Then, we deduce that $p_{\alpha_2} \equiv p_{\alpha_1}$, for all $\alpha_2 \in (0, \alpha^*)$. Thus, we conclude that the positive stationary solution does not depend on α_2. In addition, we can use the same method to prove that the positive stationary solution does not depend on η, denoted by $p_{n,R,\delta}$.

Because U_n is not trivial, we can choose η small enough such that $\underline{U}^n(x, 0) \geq \Phi_{R,\alpha_2,\eta}(x, 0)$ for all $x \in \mathbb{R}$. Then (3.42) implies that for any $R > 0$ large enough, $\delta > 0$ small enough and n large enough:

$$\liminf_{t\to+\infty} U_n\left(x_n + c_n\left(t_n + t\right) + x, t_n + t\right) \geq p_{n,R,\delta}(x), \quad \forall x \in \mathbb{R}. \tag{3.51}$$

In order to complete the proof of Claim*, it remains to check that $p_{n,R,\delta}$ is close enough to 1 as n and R are large and δ is small.

Because $p_{n,R,\delta} \leq 1$, by using a priori estimate and Arzelá-Ascoli theorem, we can extract a subsequence of the function $p_{n,R,\delta}$, as $n \to \infty$, $R \to \infty$ and $\delta \to 0$, converging locally uniformly to a stationary solution p_∞ of

$$d_1(J_1 * p_\infty - p_\infty) + c_\infty \nabla p_\infty + r_1 p_\infty(1 - p_\infty) = 0. \tag{3.52}$$

Since the map $t \mapsto \Phi_{R,\alpha_2,\eta}(x, t)$ is nondecreasing for $\alpha_2 \in (0, \alpha^*)$ and $\eta \in (0, \eta_0]$, we choose $\bar{\alpha} \in (0, \alpha^*)$ such that $p_{n,R,\delta}(x_0) \geq \Phi_{R,\bar{\alpha},\eta_0}(x_0, 0) \geq \eta_0 \mathrm{e}^{-\bar{\alpha}x_0} \cos\left(\frac{\pi x_0}{2R}\right)$. Notice that $\cos\left(\frac{\pi x_0}{2R}\right) \to 1$ locally uniformly as $R \to +\infty$, hence $p_\infty(x_0) \geq \eta_0 \mathrm{e}^{-\bar{\alpha}x_0}$ and the comparison principle in Theorem 2.1 shows that $p_\infty > 0$. To conclude we shall use the following lemma for the monostable equation:

Lemma 3.11. *Let $p = p(x)$ be a stationary of (3.52) such that $0 < p(x) \leq 1$ for all $x \in \mathbb{R}$. Then $p(x) = 1$, $\forall x \in \mathbb{R}$.*

Proof. To prove this lemma, first let $p \equiv p(x)$ be a stationary solution of (3.52) and satisfy $0 < p(x) \leq 1$ for any $x \in \mathbb{R}$. We know that the map $U(x, t) := p(x - c_\infty t)$ satisfies

$$\partial_t U - d_1(J_1 * U - U) = r_1 U(1 - U).$$

Due to $U(x, 0) = p(x) > 0$ and $c_\infty \in [0, c_U)$, it follows from Lemma 2.6 that for each $x \in \mathbb{R}$ $U(x + c_\infty t, t) \to 1$ as $t \to \infty$. In addiction, $U(x + c_\infty t, t) = p(x)$ for any $t \geq 0$ and $x \in \mathbb{R}$. Thus we complete the proof of the lemma. □

By Lemma 3.11, we can immediately infer that $p_\infty = 1$. Then, choosing R and δ such that for any $R' > 0$, $\delta' > 0$ and n large enough, we have

$$p_{n,R,\delta}(x) \geq 1 - \delta' \text{ for any } x \in (-R', R').$$

It follows from (3.51) that (3.28) holds true. Thus we complete the proof of Claim*. □

4. Fast predator case

In this section, we mainly explore the behavior of the solution of system (1.1) when the prey cannot exceed the predator.

Theorem 4.1. *(Fast predator) Assume* (J1), (J3)-(J4) *and* (H1) *hold.* u_0, v_0 *be two given nontrivial compactly supported functions such that* $(u_0, v_0) \in H$. *In addition, we further assume that* $d_1 > r_1 + b\hat{r}_2 + \frac{b}{2} + \beta b\hat{r}_2$ *and* $d_2 > \beta b - a + \beta b\hat{r}_2 + \frac{b}{2}$.

If $c_V \geq c_U$, *then the solution* $(U, V) \equiv (U(x,t), V(x,t))$ *of* (1.1) *with initial data* (u_0, v_0) *satisfies the following statements.*

(i) For any $c > c_V$, *then*

$$\lim_{t\to\infty} \sup_{|x|>ct} V(x,t) = 0.$$

(ii) For any $c > c_U$, *then*

$$\lim_{t\to\infty} \sup_{|x|>ct} U(x,t) = 0.$$

(iii) There exists $\varepsilon > 0$ *such that for each* $c \in [0, c_U)$, *one has*

$$\liminf_{t\to\infty} \inf_{|x|\leq ct} V(x,t) \geq \varepsilon,$$

$$\limsup_{t\to\infty} \sup_{|x|\leq ct} U(x,t) \leq 1-\varepsilon \text{ and } \liminf_{t\to\infty} \inf_{|x|\leq ct} U(x,t) \geq \varepsilon.$$

Proof. First, we prove the statement (i). In particular, for any $c > c_V$, there exists some $0 < \eta < 1$ such that

$$\hat{c} = \inf_{0<\lambda<+\infty} \frac{d_2[\int_{\mathbb{R}} J_2(x)e^{-\lambda x}\mathrm{d}x - 1] + \kappa - a}{\lambda} < c,$$

where $\kappa := \frac{\beta b\eta}{1+\alpha\eta}$ and $\kappa > a$. Then there is some $\lambda > 0$ such that

$$\tilde{\Delta}_2(\lambda, c) = d_2 \int_{\mathbb{R}} J(y)e^{\lambda y}\mathrm{d}y - d_2 - c\lambda + \kappa - a = 0.$$

Using (3.2), there exists $t_1 > 0$ such that $\forall t \geq t_1$,

$$\sup_{t\geq t_1} \sup_{|x|\geq ct} U(x,t) \leq \eta.$$

Then the function $\overline{V}_2 = Ae^{-\lambda(x-ct)}$ for any constant $A > 0$ satisfies

$$\begin{aligned}
&\partial_t \overline{V}_2 - d_2 \int_{\mathbb{R}} J(y)\overline{V}_2(x-y,t)\mathrm{d}y + d_2\overline{V}_2 - \overline{V}_2(\kappa - a)\\
=&c\lambda\overline{V}_2 - d_2 \int_{\mathbb{R}} J(y)\mathrm{e}^{\lambda y}\overline{V}_2\mathrm{d}y + d_2\overline{V}_2 - \overline{V}_2(\kappa - a)\\
=&\overline{V}_2\left(c\lambda - d_2\int_{\mathbb{R}} J(y)\mathrm{e}^{\lambda y}\mathrm{d}y + d_2 - \kappa + a\right)\\
=&\tilde{\Delta}_2(\lambda,c)\overline{V}_2\\
=&0.
\end{aligned}$$

Thus, $\overline{V}_2$ is a sub-solution of (1.1) for any $t \geq t_1$ and $x \geq ct$. Furthermore, according to boundedness, there is a sufficiently large A such that

$$V(x,t) \leq A \leq \overline{V}_2(x,t), \quad \forall t \geq t_1, |x| = ct.$$

It follows from Lemma 2.9 that the equation $\Delta_2(\lambda,c) = 0$ has two positive roots $\lambda_1 = \lambda_1(c)$, $\lambda_2 = \lambda_2(c)$ and $0 < \lambda_1 < \lambda_2 < +\infty$ for any $c > c_V$. Moreover, there is $\tilde{\lambda} \in (\lambda_1, \lambda_2)$ such that $\Delta_2(\tilde{\lambda}, c) < 0$. According to the properties of $\tilde{\Delta}_2(\lambda,c)$ and $\Delta_2(\lambda,c)$, we can take $0 < \lambda < \lambda_1 < \tilde{\lambda}$. Using (3.3) and choosing A large enough, we have

$$V(x,t_1) \leq \overline{V}(x,t_1) \leq \overline{V}_2(x,t_1), \text{ for all } x \in \mathbb{R}.$$

Applying the comparison principle, one has

$$\lim_{t\to\infty} \sup_{|x|>c't} V(x,t) = 0, \quad \forall c' > c.$$

Since c can be chosen arbitrarily close to c_V, this means that V does not spread faster than the speed c_V. Thus, we complete the proof of statement (i).

The proofs of statements (ii) and (iii) are similar to Propositions 3.2 and 3.5, respectively. Hence, we omit it. This completes the proof of Theorem 4.1. □

5. Asymptotic behavior of the predator and the prey

This section is mainly concerned with the analysis of the long time behavior of the predator and the prey, proving that they will eventually coexist.

Theorem 5.1. *In addition to* (J1), (J3)-(J4) *and* (H1) *hold, suppose that system* (1.1) *has a unique positive equilibrium point* (u^*, v^*). *We further assume that* $r_1 > b\alpha\hat{r}_2 + \frac{1}{2}b\alpha$, $2a > \beta b\alpha$, $d_1 > r_1 + b\hat{r}_2 + \frac{b}{2} + \beta b\hat{r}_2$ *and* $d_2 > \beta b - a + \beta b\hat{r}_2 + \frac{b}{2}$. *Let* $(U,V) \equiv (U(x,t), V(x,t))$ *be a solution of system* (1.1) *with nontrivial compactly supported initial data* $(u_0, v_0) \in H$ *with* $u_0 \not\equiv 0$ *and* $v_0 \not\equiv 0$. *Then for each* $c \in [0, \min\{c_V, c_U\})$ *one has*

$$\lim_{t\to\infty} \sup_{|x|\leq ct} (|U(x,t) - u^*| + |V(x,t) - v^*|) = 0.$$

Proof. We proceed by contradiction. Assume that there exist $c \in [0, \min\{c_V, c_U\})$, a sequence $\{(x_k, t_k)\}_{k\geq 0} \subset \mathbb{R} \times (0, \infty)$ such that $t_k \to \infty$ and $\delta > 0$ such that for all $k \geq 0$:

$$|x_k| \leq ct_k \text{ and } |U(x_k, t_k) - u^*| + |V(x_k, t_k) - v^*| \geq \delta. \tag{5.1}$$

Consider the sequence of function (U_k, V_k) defined by

$$(U_k, V_k)(x, t) = (U, V)(x + x_k, t + t_k).$$

We fix $c' > 0$ such that $c < c' < c_V$. From Theorems 3.1 and 4.1, there exist $A > 0$ large enough and $\varepsilon > 0$ small enough such that for $k \geq 0$, $x \in \mathbb{R}$ and $t \in \mathbb{R}$, we obtain that

$$t + t_k \geq A \text{ and } |x| \leq c't + (c' - c)t_k \Rightarrow \begin{cases} \varepsilon \leq U_k(x, t) \leq 1, \\ \varepsilon \leq V_k(x, t) \leq \hat{r}_2. \end{cases} \tag{5.2}$$

By some priori estimates, Arzelá-Ascoli theorem and the diagonal extraction process, we can extract a subsequence such that

$$(U_k, V_k)(x, t) \to (U_\infty, V_\infty)(x, t) \text{ locally uniformly for } (x, t) \in \mathbb{R} \times \mathbb{R}, \tag{5.3}$$

where (U_∞, V_∞) is a bounded entire solution of (1.1). Moreover, by (5.2), the function (U_∞, V_∞) satisfies

$$\inf_{(x,t)\in\mathbb{R}\times\mathbb{R}} U_\infty(x, t) > 0 \text{ and } \inf_{(x,t)\in\mathbb{R}\times\mathbb{R}} V_\infty(x, t) > 0,$$
$$\sup_{(x,t)\in\mathbb{R}\times\mathbb{R}} U_\infty(x, t) \leq 1,$$

where (5.1) ensures that

$$|U_\infty(0, 0) - u^*| + |V_\infty(0, 0) - v^*| \geq \delta. \tag{5.4}$$

On the other hand, define

$$\tilde{W}(U, V)(t) = 2\int_{\mathbb{R}} \left[\beta u^* g\left(\frac{U(x, t)}{u^*}\right) + v^* g\left(\frac{V(x, t)}{v^*}\right)\right] \mathrm{d}x,$$

where $g(z) = z - 1 - \ln z$, $z > 0$ and $g(z) \geq 0$ for all $z > 0$. Since $u_0 \not\equiv 0$, $v_0 \not\equiv 0$ and $(u_0, v_0) \in H$, then the solution (U, V) of system (1.1) satisfies

$$0 < U(x, t) \leq 1, \quad 0 < V(x, t) \leq \hat{r}_2, \quad \forall (x, t) \in \mathbb{R} \times \mathbb{R}.$$

Thus $\tilde{W}(U, V)(t)$ is well-defined. The derivative of $\tilde{W}(U, V)(t)$ along solutions of system (1.1) is obtained as follows:

$$
\begin{aligned}
&\frac{\mathrm{d}\tilde{W}(U,V)}{\mathrm{d}t}(t)\\
=&2\int\limits_{\mathbb{R}}\left[\beta\left(1-\frac{u^*}{U(x,t)}\right)\frac{\partial U}{\partial t}(x,t)+\left(1-\frac{v^*}{V(x,t)}\right)\frac{\partial V}{\partial t}(x,t)\right]\mathrm{d}x\\
=&2\int\limits_{\mathbb{R}}\left\{\beta\left(1-\frac{u^*}{U(x,t)}\right)\left[d_1(J_1*U-U)(x,t)+U(x,t)\left(r_1-r_1U(x,t)-\frac{bV(x,t)}{1+\alpha U(x,t)}\right)\right]\right.\\
&\left.+\left(1-\frac{v^*}{V(x,t)}\right)\left[d_2(J_2*V-V)(x,t)+V(x,t)\left(\frac{\beta bU(x,t)}{1+\alpha U(x,t)}-a-aV(x,t)\right)\right]\right\}\mathrm{d}x\\
=&2\int\limits_{\mathbb{R}}\left\{\beta\left(1-\frac{u^*}{U(x,t)}\right)\left[d_1(J_1*U-U)(x,t)+U(x,t)\Big(r_1(u^*-U(x,t))\right.\right.\\
&\left.+\frac{bv^*}{1+\alpha u^*}-\frac{bV(x,t)}{1+\alpha U(x,t)}\Big)\right]+\left(1-\frac{v^*}{V(x,t)}\right)\left[d_2(J_2*V-V)(x,t)\right.\\
&\left.\left.+V(x,t)\Big(\frac{\beta bU(x,t)}{1+\alpha U(x,t)}-\frac{\beta bu^*}{1+\alpha u^*}+a(v^*-V(x,t))\Big)\right]\right\}\mathrm{d}x\\
=&2\int\limits_{\mathbb{R}}\left\{\beta\left(1-\frac{u^*}{U(x,t)}\right)[d_1(J_1*U-U)(x,t)]+\left(1-\frac{v^*}{V(x,t)}\right)[d_2(J_2*V-V)(x,t)]\right.\\
&-r_1\beta(U(x,t)-u^*)^2-a(V(x,t)-v^*)^2+\frac{\beta b\alpha u^*(U(x,t)-u^*)(v*-V(x,t))}{(1+\alpha u^*)(1+\alpha U(x,t))}\\
&\left.+\frac{\beta b\alpha v^*(U(x,t)-u^*)^2}{(1+\alpha u^*)(1+\alpha U(x,t))}\right\}\mathrm{d}x\\
\le&2\int\limits_{\mathbb{R}}\left\{\beta\left(1-\frac{u^*}{U(x,t)}\right)\left[d_1\int\limits_{\mathbb{R}}J_1(x-y)U(y,t)\mathrm{d}y-d_1\frac{U(x,t)}{u^*}\int\limits_{\mathbb{R}}J_1(x-y)u^*\mathrm{d}y)\right]\right.\\
&+\left(1-\frac{v^*}{V(x,t)}\right)\left[d_2\int\limits_{\mathbb{R}}J_2(x-y)V(y,t)\mathrm{d}y-d_2\frac{V(x,t)}{v^*}\int\limits_{\mathbb{R}}J_2(x-y)v^*\mathrm{d}y)\right]\\
&+\left(\frac{\beta b\alpha v^*}{(1+\alpha u^*)(1+\alpha U(x,t))}+\frac{\beta b\alpha u^*}{2(1+\alpha u^*)(1+\alpha U(x,t))}-r_1\beta\right)(U(x,t)-u^*)^2\\
&\left.+\left(\frac{\beta b\alpha u^*}{2(1+\alpha u^*)(1+\alpha U(x,t))}-a\right)(V(x,t)-v^*)^2\right\}\mathrm{d}x\\
\le&2\int\limits_{\mathbb{R}}\left[d_1\beta\left(1-\frac{u^*}{U(x,t)}\right)\int\limits_{\mathbb{R}}J_1(x-y)U(y,t)\mathrm{d}y+d_1\beta\left(1-\frac{U(x,t)}{u^*}\right)\int\limits_{\mathbb{R}}J_1(x-y)u^*\mathrm{d}y\right.
\end{aligned}
$$

$$+ d_2\left(1-\frac{v^*}{V(x,t)}\right)\int\limits_{\mathbb{R}} J_2(x-y)V(y,t)\mathrm{d}y + d_2\left(1-\frac{V(x,t)}{v^*}\right)\int\limits_{\mathbb{R}} J_2(x-y)v^*\mathrm{d}y$$

$$+\left(\beta b\alpha\hat{r}_2+\frac{1}{2}\beta b\alpha - r_1\beta\right)(U(x,t)-u^*)^2+\left(\frac{1}{2}\beta b\alpha - a\right)(V(x,t)-v^*)^2\Bigg]\mathrm{d}x. \tag{5.5}$$

The third equality of the above formula is based on the fact that $E^* := (u^*, v^*)$ is the equilibrium of system (1.1), namely

$$\begin{cases} r_1u^* + \frac{bv^*}{1+\alpha u^*} = r_1, \\ \frac{\beta bu^*}{1+\alpha u^*} - av^* = a. \end{cases}$$

According to the assumptions, we have that

$$b\alpha\hat{r}_2 + \frac{1}{2}b\alpha - r_1 < 0 \text{ and } \frac{1}{2}\beta b\alpha - a < 0. \tag{5.6}$$

Note that

$$\begin{aligned}
&2\beta d_1\int\limits_{\mathbb{R}}\left[\left(1-\frac{u^*}{U(x,t)}\right)\int\limits_{\mathbb{R}} J_1(x-y)U(y,t)\mathrm{d}y+\left(1-\frac{U(x,t)}{u^*}\right)\int\limits_{\mathbb{R}} J_1(x-y)u^*\mathrm{d}y\right]\mathrm{d}x\\
=&2\beta d_1\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} J_1(x-y)u^*\left[\left(1-\frac{u^*}{U(x,t)}\right)\frac{U(y,t)}{u^*}+\left(1-\frac{U(x,t)}{u^*}\right)\right]\mathrm{d}y\mathrm{d}x\\
=&2\beta d_1\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} J_1(x-y)u^*\left[1-\frac{U(x,t)}{u^*}+\frac{U(y,t)}{u^*}-\frac{U(y,t)}{U(x,t)}\right]\mathrm{d}y\mathrm{d}x\\
=&\beta d_1\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} J_1(x-y)u^*\left[1-\frac{U(x,t)}{u^*}+\frac{U(y,t)}{u^*}-\frac{U(y,t)}{U(x,t)}\right]\mathrm{d}y\mathrm{d}x\\
&+\beta d_1\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} J_1(y-x)u^*\left[1-\frac{U(y,t)}{u^*}+\frac{U(x,t)}{u^*}-\frac{U(x,t)}{U(y,t)}\right]\mathrm{d}x\mathrm{d}y\\
=&\beta d_1\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} J_1(x-y)u^*\left[1-\frac{U(x,t)}{u^*}+\frac{U(y,t)}{u^*}-\frac{U(y,t)}{U(x,t)}\right]\mathrm{d}y\mathrm{d}x\\
&+\beta d_1\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} J_1(x-y)u^*\left[1-\frac{U(y,t)}{u^*}+\frac{U(x,t)}{u^*}-\frac{U(x,t)}{U(y,t)}\right]\mathrm{d}y\mathrm{d}x\\
=&\beta d_1\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} J_1(x-y)u^*\left[2-\frac{U(y,t)}{U(x,t)}-\frac{U(x,t)}{U(y,t)}\right]\mathrm{d}y\mathrm{d}x.
\end{aligned} \tag{5.7}$$

Similarly, one has

$$2d_2 \int_{\mathbb{R}} \left[\left(1 - \frac{v^*}{V(x,t)}\right) \int_{\mathbb{R}} J_2(x-y)V(y,t)\mathrm{d}y + \left(1 - \frac{V(x,t)}{v^*}\right) \int_{\mathbb{R}} J_2(x-y)v^*\mathrm{d}y \right]$$
$$= d_2 \int_{\mathbb{R}} \int_{\mathbb{R}} J_2(x-y)v^* \left[2 - \frac{V(y,t)}{V(x,t)} - \frac{V(x,t)}{V(y,t)}\right] \mathrm{d}y\mathrm{d}x. \tag{5.8}$$

Substitute (5.7) and (5.8) to (5.5), it follows from (5.6) that $\frac{\mathrm{d}\tilde{W}(U,V)}{\mathrm{d}t}(t) \le 0$. Recall that $0 < U \le 1$, $0 < V \le \hat{r}_2$, thus $\tilde{W}(U,V)(t)$ is non-increasing on t and bounded. There exists a constant M_1 such that

$$M_1 \le \tilde{W}(U_k, V_k)(t) = \tilde{W}(U,V)(t+t_k) \le \tilde{W}(U,V)(t).$$

Thus, there exists a constant $\tilde{w}$ such that

$$\lim_{k\to\infty} \tilde{W}(U_k, V_k)(t) = \tilde{w}.$$

Together with (5.3), it follows that $\tilde{W}(U_\infty, V_\infty)(t) \equiv \tilde{w}$. Note that $\frac{\mathrm{d}\tilde{W}(U,V)}{\mathrm{d}t}(t) = 0$ if and only if $U(x,t) = u^*$, $V(x,t) = v^*$, respectively. Thus $(U_\infty, V_\infty) = (u^*, v^*)$ which contradicts (5.4). This completes the proof of Theorem 5.1. □

6. Numerical simulation

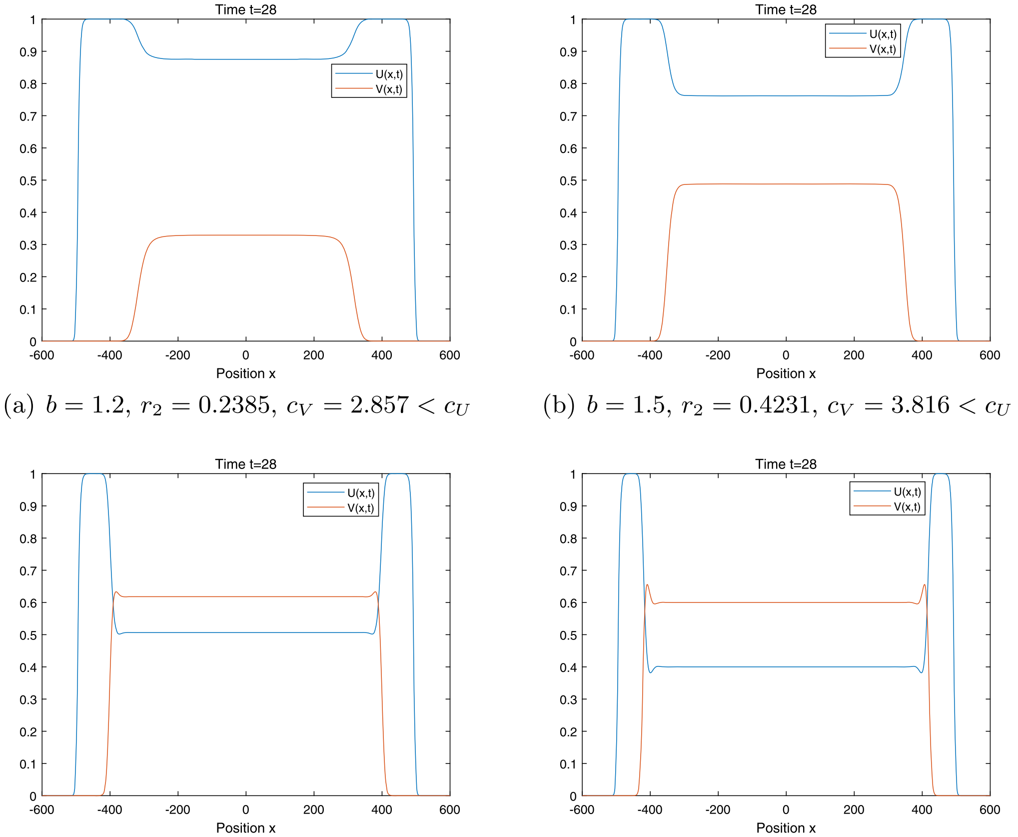
In this section, we perform some numerical simulations to illustrate the two cases stated in Theorems 3.1 and 4.1, respectively.

We choose the kernel function $J_i(x) = J_\varrho(x) = \frac{1}{\sqrt{2\pi\varrho^2}} e^{\frac{-x^2}{2\varrho^2}}$ $(i = 1, 2)$ and $\varrho = 1$. Furthermore, the initial functions are chosen as

$$u_0(x) = \begin{cases} 0.3, & |x| \le 150, \\ \frac{1}{500}(300 - |x|), & 150 < |x| \le 300, \\ 0, & |x| > 300, \end{cases} \qquad v_0(x) = \frac{1}{3}u_0(x). \tag{6.1}$$

For the case when the predator spreads slower than the prey, that is, $c_V < c_U$, we fix $a = 0.5$, $\alpha = 0.3$, $\beta = 0.8$, $d_1 = d_2 = 17$, $r_1 = 2.5$ and vary the parameters b and $r_2 = \frac{\beta b}{1+\alpha} - a$. From the definition of c_U, we get that $c_U = 9.527$. When the compactly supported disturbance of the predator and the prey is initially located at the center of the frame, spatial invasions of the predator and the prey arise in both sides of the domain. This phenomenon is depicted in Figs. 1 and 2. Specifically, with the help of MATLAB, we obtain the snapshots of the solution of system (1.1) with initial value (6.1) at time $t = 28$ (see Fig. 1) and at three different times (see Fig. 2).

Fig. 1 shows that the predator invades the environment slower than the prey and the propagation occurs in two separate processes. More specifically, the prey first invades the environment and continues to multiply, making its population density reach the environmental capacity. With the invasion of the predator, the predator will capture the prey, so that the density of the prey decreases and the density of the predator increases, and finally the predator and the prey coexist. It can be seen from Fig. 1 that when the predation rate b increases, the solution of system (1.1) will produce damped oscillations, and the greater the predation rate, the larger the magnitude of the



(a) $b = 1.2,\ r_2 = 0.2385,\ c_V = 2.857 < c_U$ (b) $b = 1.5,\ r_2 = 0.4231,\ c_V = 3.816 < c_U$

(c) $b = 2.3,\ r_2 = 0.9154,\ c_V = 5.651 < c_U$ (d) $b = 2.8,\ r_2 = 1.2231,\ c_V = 6.559 < c_U$

Fig. 1. The spatial distributions of the prey U and the predator V at time $t = 28$ obtained for various different values for the parameters b and r_2.

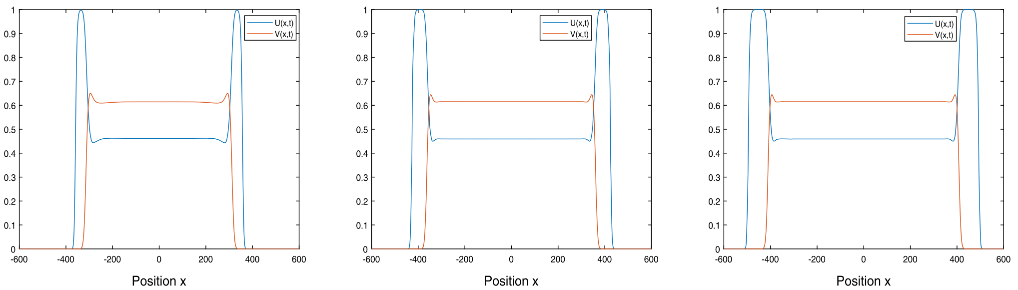
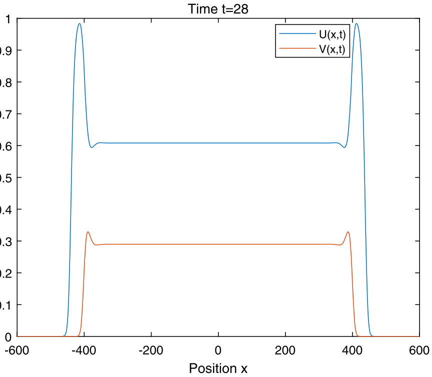
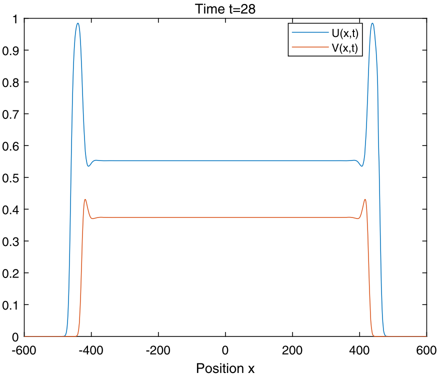


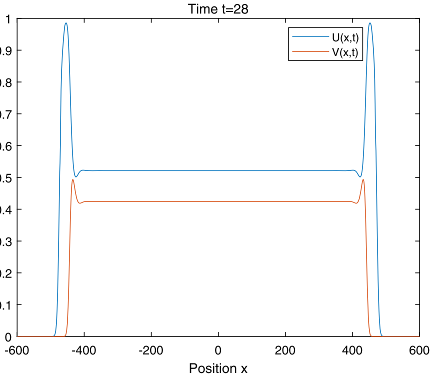
Fig. 2. The spatial distributions of the prey U and the predator V at three different times (increasing from left to right) with $b = 2.5$, $r_2 = 1.0385$.



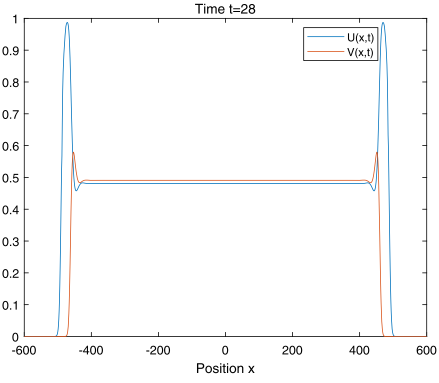
(a) $b = 1.8$, $r_1 = r_2 = 1.2545$, $c_V = c_U = 6.645$



(b) $b = 2.1$, $r_1 = r_2 = 1.6636$, $c_V = c_U = 7.693$



(c) $b = 2.3$, $r_1 = r_2 = 1.9364$, $c_V = c_U = 8.327$



(d) $b = 2.6$, $r_1 = r_2 = 2.3455$, $c_V = c_U = 9.211$

Fig. 3. The spatial distributions of the prey U and the predator V at time $t = 28$ obtained for various different values for the parameters b, r_1 and r_2.

oscillations. Fig. 2 shows the spatial distributions of the prey and the predator at three different times with the same choices of the parameters as above and with $b = 2.5$.

For the case when the predator spreads faster than the prey, we fix $a = 1.2$, $\alpha = 0.1$, $\beta = 1.5$, $d_1 = d_2 = 17$ and vary the parameters b, r_1. First, we consider the case where the speeds of the predator and the prey are the same, that is, $c_V = c_U$. We consider this situation because when the predator spreads faster than the prey, the predator cannot survive without food. From the definitions of c_V, c_U and r_2, we can change the value of r_1 to ensure that $c_V = c_U$ when the value of b changes. Figs. 3 and 4 show that when the population of the predator grows fast enough to catch up with the prey, the two species spread nearly simultaneously. In addition, it can be seen from Figs. 3 and 4 that there are some spatial gap between the front of the prey and that of the predator, but the gap shall be of order $o(t)$ at most.

Second, we consider the case where the spreading speed of the predator is faster than the prey, that is, $c_V > c_U$. In addition to the same choice of the parameters as above, we further fix $r_1 = 1$. From the definition of c_U, we get that $c_U = 5.913$. Based on our theoretical results, namely, Theorems 4.1 and 5.1, we can see from Figs. 5 and 6 that the system spreads in fact just as fast as the prey. This spreading phenomenon is similar to the first case $c_V = c_U$ of fast predator.

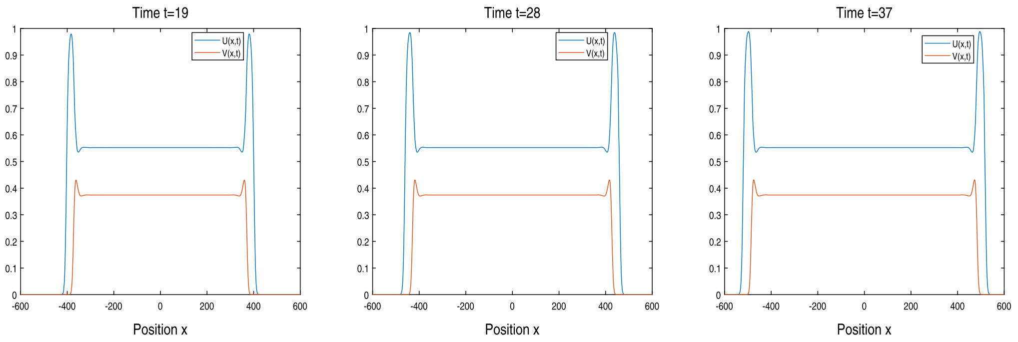
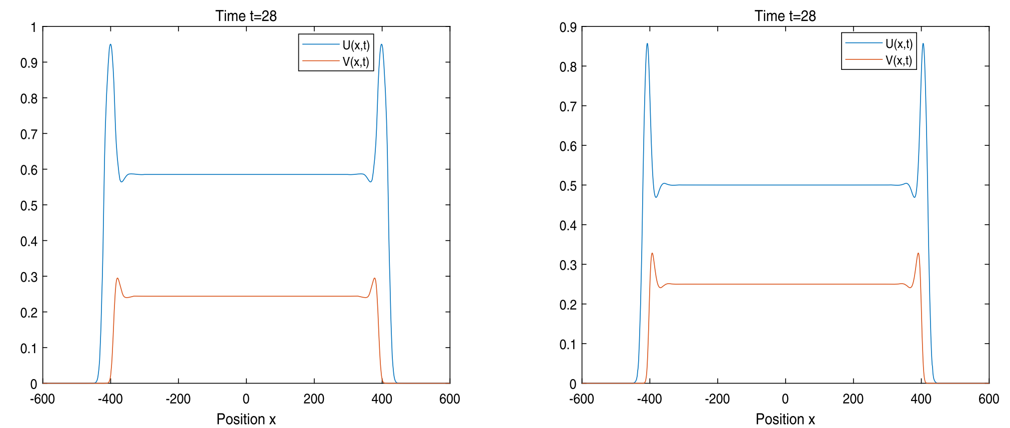
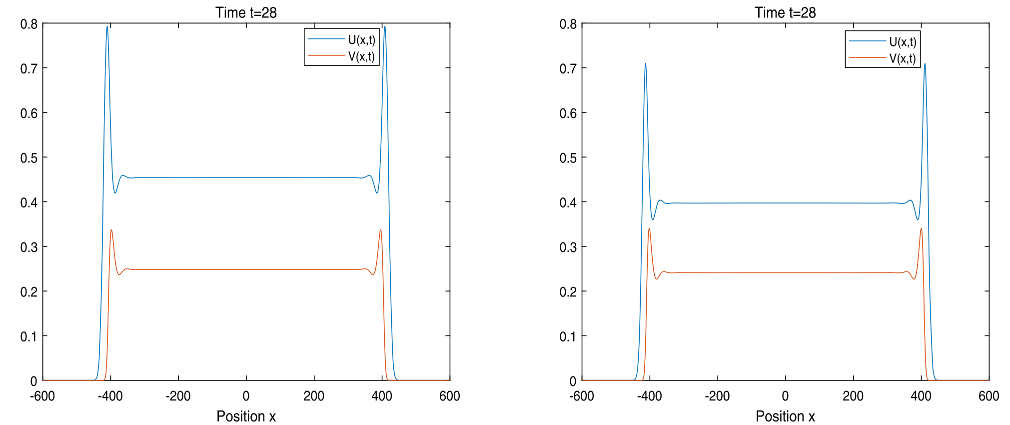


Fig. 4. The spatial distributions of the prey U and the predator V at the three different times (increasing from left to right) with $b = 2.1$, $r_1 = r_2 = 1.6636$.



(a) $b = 1.8,\ r_2 = 1.2545,\ c_V = 6.645 > c_U$ (b) $b = 2.1,\ r_2 = 1.6636,\ c_V = 7.693 > c_U$



(c) $b = 2.3,\ r_2 = 1.9364,\ c_V = 8.327 > c_U$ (d) $b = 2.6,\ r_2 = 2.3455,\ c_V = 9.211 > c_U$

Fig. 5. The spatial distributions of the prey U and the predator V at a time $t = 28$ obtained for various different values for the parameters b and r_2.

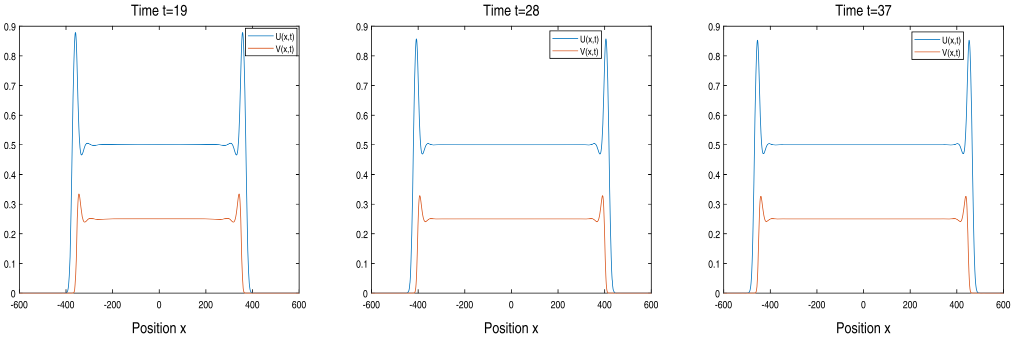


Fig. 6. The spatial distributions of the prey U and the predator V at three different times (increasing from left to right) with $b = 2.1$, $r_2 = 1.6636$.

Moreover, Fig. 5 shows that when the predation rate b increases, the solution of system (1.1) will produce damped oscillations, and the greater the predation rate, the larger the magnitude of the oscillations. Fig. 6 shows the spatial distributions of the prey and the predator at three different times.

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