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THE PERSISTENCE OF SOLUTIONS IN A NONLOCAL PREDATOR-PREY SYSTEM WITH A SHIFTING HABITAT

Min ZHAO (赵敏)*
School of Science, Tianjin Chengjian University, Tianjin 300384, China
E-mail: *minzhao9216@163.com*

Rong YUAN (袁荣)
Laboratory of Mathematics and Complex Systems (Ministry of Education), School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China
E-mail: *ryuan@bnu.edu.cn*

Abstract In this paper, we mainly study the propagation properties of a nonlocal dispersal predator-prey system in a shifting environment. It is known that Choi *et al.* [J Differ Equ, 2021, 302: 807–853] studied the persistence or extinction of the prey and of the predator separately in various moving frames. In particular, they achieved a complete picture in the local diffusion case. However, the question of the persistence of the prey and of the predator in some intermediate moving frames in the nonlocal diffusion case was left open in Choi *et al.*'s paper. By using some *a prior* estimates, the Arzelà-Ascoli theorem and a diagonal extraction process, we can extend and improve the main results of Choi *et al.* to achieve a complete picture in the nonlocal diffusion case.

Key words predator-prey system; persistence; nonlocal dispersal; shifting environment

2020 MR Subject Classification 35K57; 35K55; 35B40; 92D25

1 Introduction

In recent years, in addition to seasonal and regional differences, climate change caused by global warming, industrialization and overdevelopment have had a huge impact on the habitats of biological species [37]. The habitats of species will move in both time and space on account of climate change [36, 37]. A simple pattern for measuring climate change is the shifting of environment quality with constant speed. This translates into the shifting of habitat quality, which is reflected in the shifting of the growth rate for a species [39]. At present, many scholars have devoted themselves to studying these topics; see [3, 5, 10, 12, 13, 16, 22, 25, 26, 28–31, 34, 42].

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*Corresponding author

Random diffusion describes a process whereby organisms can only move to their surrounding neighborhoods [1, 19]. Based on this property of random diffusion, we can describe the dynamics of random dispersal through reaction-diffusion model. There has been much research on the spreading speeds and forced waves of reaction-diffusion systems in a shifting environment. For the case of the scalar equation, Bouhours *et al.* [6], Hu *et al.* [15] and Li *et al.* [25] explored the conditions for the extinction and the persistence as well as the spatial-temporal dynamics of a species with a shifting habitat edge. Fang *et al.* [11] and Vo [31] investigated the propagation dynamics of a reaction-diffusion equation in a time-periodic shifting environment. Berestycki *et al.* [3–5] established the existence of forced waves of reaction-diffusion equations for population dynamics with a shifting habitat. Hu *et al.* [16] established the existence of an extinction wave in the Fisher equation with a shifting habitat. For the case of a competition model, Potapov and Lewis [28] considered a Lotka-Volterra competition model in a domain with a moving range boundary, by which they obtained a critical patch size for each species to persist and spread. Yuan *et al.* [39] and Zhang *et al.* [40] studied the persistence versus extinction for two competing species under climate change. Dong *et al.* [9] investigated the existence of forced waves in a Lotka-Volterra competition-diffusion model with a shifting habitat. Berestycki *et al.* [2] investigated the persistence of two species and the gap formation of a Lotka-Volterra competition model. For the case of a cooperative model, Yang *et al.* [38] considered the existence and asymptotics of forced wave solutions in a Lotka-Volterra cooperative model under climate change. We also refer the readers to some literature considering forced waves in domains with a free boundary [14, 22].

In addition to random diffusion, nonlocal dispersal is a more reasonable process for some species to travel for some distance, and their movements and interactions may occur between non-adjacent spatial locations [1, 17, 19, 20, 23, 44]. The widespread long-distance dispersal or nonlocal internal interactions are usually modeled by an appropriate integral operator, such as $\int_{\mathbb{R}} J(x-y)[u(y)-u(x)]\mathrm{d}y$. Concerning the study of the effects of the climate change in nonlocal dispersal, the work of Coville [8], Leenheer *et al.* [21], Li *et al.* [27], and Wang *et al.* [32] studied the persistence criterion and the existence and uniqueness, as well as the stability of forced waves for the scalar nonlocal dispersal population model in a shifting environment. In addition, Zhang *et al.* [41] explored the propagation dynamics of a nonlocal dispersal Fisher-KPP equation in a time-periodic shifting habitat. For the case of a competition system, Wu *et al.* [37] studied the spatio-temporal spreading dynamics of a Lotka-Volterra competition model with nonlocal dispersal under a shifting environment. Wang *et al.* [33] and Wang *et al.* [35] investigated the existence of forced waves and gap formations for the lattice and continuous Lotka-Volterra competition models with nonlocal dispersal and shifting habitats, respectively. Bao *et al.* [1] studied the traveling wave solutions of Lotka-Volterra competition systems with nonlocal dispersal in periodic habitats. For the case of the prey-predator system, Choi *et al.* [7] studied the persistence of a species in a predator-prey system with climate change and either nonlocal or local dispersal.

Due to the lack of a comparison principle and the issue of the compactness of the set of solutions with bounded initial data, there is less work on predator-prey systems with nonlocal dispersal in shifting environments. Motivated by the aforementioned works, we would like to extend and improve the work of Choi *et al.* [7] in order to deal with the spreading population dynamics of predator-prey species in some intermediate moving frames with nonlocal dispersal under a shifting habitat. In this paper, we consider the following predator-prey model with nonlocal dispersal proposed by Choi *et al.* [7]:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = d_1[(J_1 * u)(x,t) - u(x,t)] + r_1 u(x,t)[\alpha(x-st) - u(x,t) - av(x,t)], \\ \dfrac{\partial v}{\partial t}(x,t) = d_2[(J_2 * v)(x,t) - v(x,t)] + r_2 v(x,t)[-1 + bu(x,t) - v(x,t)]. \end{cases} \tag{1.1}$$

Here $x \in \mathbb{R}, t > 0$ and r_1, r_2, a, b are all positive constants, and $u(x,t)$ and $v(x,t)$ are the population densities of the prey and predator species at spatial position $x \in \mathbb{R}$ and time $t > 0$, respectively. The dynamics of the prey population follow a logistic growth, which depends on a shifting habitat with a fixed speed $s > 0$. Parameters r_1 and r_2 denote the intrinsic growth rates. The constant a denotes the predation rate and b denotes the biomass conversion rate. $d_1 > 0$ and $d_2 > 0$ are the diffusion coefficients for prey and predator species, respectively. The term $J_i * w - w$ describes the spatial dispersal process and

$$(J_i * w)(x,t) - w(x,t) = \int_{\mathbb{R}} J_i(x-y)w(y,t)\mathrm{d}y - w(x,t), \quad i = 1,2,$$

where the symbol $*$ denotes the convolution product for the spatial variable. Here we assume that the kernel function $J_i : \mathbb{R} \to \mathbb{R} (i = 1,2)$ is continuous and satisfies the following properties:

(J1) $J_i(x) = J_i(-x) \geq 0$ for any $x \in \mathbb{R}$ and $\int_{\mathbb{R}} J_i(x)\mathrm{d}x = 1, i = 1,2$;

(J2) $J_i \in C^1(\mathbb{R})$ and J_i is compactly supported, $i = 1,2$.

The function $\alpha(\cdot)$ models climate change, which depends on a shifting variable, and throughout the paper we assume that it satisfies the following properties:

(α_1) $\alpha(\cdot) \in C^1(\mathbb{R})$ and nondecreasing in $\mathbb{R}$;

(α_2) $-\infty < \alpha(-\infty) < 0 < \alpha(\infty) < \infty$; furthermore, we choose $\alpha(\infty) = 1$, without loss of generality (up to a rescaling);

(α_3) the derivative of $\alpha(\cdot)$ is bounded in $\mathbb{R}$.

It is clear that the shifting environment may be divided into a favorable region $\{x \in \mathbb{R} : \alpha(x-st) > 0\}$ and an unfavorable region $\{x \in \mathbb{R} : \alpha(x-st) \leq 0\}$, both shifting with a speed of $s > 0$. The non-decreasing property of $\alpha(\cdot)$ assumes that the environment gets worse as time goes on, and the negativity of $\alpha(-\infty)$ accounts for a scenario in which the environment is shifting to a very severe level in the unfavorable region. In assumption (α_3), we only require the derivative of $\alpha(\cdot)$ to be bounded instead of uniformly continuous, mainly on account of more dramatic effects of global warming, including changes in the frequency of severe rainstorms, hurricanes and other climatic disasters, which may have a short-term consequences on the survival and reproduction of local or regional species [18, 24].

System (1.1) is supplemented by the initial condition

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \mathbb{R}, \tag{1.2}$$

where $u_0(x), v_0(x)$ are bounded nonnegative functions with a nonempty compact support.

Throughout this work, we give the following assumption about the parameter:

(H1) $b > 1$.

This ensures that the amount of prey is sufficient to maintain the positive density of the predators. From system (1.1), we can see that the predator cannot survive without the prey.

It is known that Choi *et al.* [7] explored the propagation properties of the predator and the prey of system (1.1) in two different situations, namely, where the prey is faster than the predator, in the sense that its maximal speed s^* is larger than the maximal speed s_* of the predator, and the situation where $s^* \le s_*$, i.e., the predator is faster than the prey. Here, the speed s^* represents the maximum speed of the prey in the "favorable" environment and without a predator; the speed s_* denotes the maximum speed of the predator when the prey density is at saturation; see Section 2 for details. For the persistence of system (1.1), Choi *et al.* showed that the predator and the prey are persistent in cases $s < s^{**}$ ($s^{**} < s^*$) and $s < \underline{s}^* = \min\{s^{**}, s_{**}\}$ ($s_{**} < s_*$), respectively. Here, s^{**} is the speed of the prey in a favorable environment when there is a maximal amount of predators, and s_{**} is the speed of the predator when there is a minimal amount of prey. However, the proof of the persistence of the predator and the prey of system (1.1) with speed s and $\min\{s_*, s^*\} > s$ remains an open question. In this paper, we extend and improve the main results in Choi *et al.* [7] to show that both species always persist and achieve a complete picture of the spreading dynamics of (1.1). Inspired by the work of Choi *et al.* [7] and Zhang *et al.* [43], we use some *a prior* estimates, the Arzelà-Ascoli theorem and a diagonal extraction process to perform various limiting arguments. We conclude that under certain conditions, the predator and the prey persist.

The rest of this paper is organized as follows: in the next section, we establish some preliminary results. In Section 3, we mainly consider the persistence of the prey u of system (1.1) with an initial value (1.2) in the moving frames with speeds between s and $\underline{s} = \min\{s^*, s_*\} > s$. In Section 4, we show the persistence of the predator v in the moving frames with speeds between s and $\underline{s} = \min\{s^*, s_*\} > s$.

2 Preliminaries

In this section, we mainly introduce some preliminaries and recall some results for spreading speeds.

First, we define that

$$X = \{w(x) |\ w(x) : \mathbb{R} \to \mathbb{R} \text{ is bounded and uniformly continuous}\},$$

with the norm

$$\|w\|_X = \sup_{x \in \mathbb{R}} |w(x)|.$$

Then $(X, \|\cdot\|_X)$ is a Banach space. Furthermore, for any constant $d > 0$, let

$$X_d = \{w \in X : 0 \le w(x) \le d,\ \forall x \in \mathbb{R}\}.$$

Set the order of the space $X^2 = X \times X$ as

$$\underline{w} \le \overline{w} \Leftrightarrow \underline{w}_i(x) \le \overline{w}_i(x), \quad x \in \mathbb{R}, i = 1, 2$$

for any $\underline{w} = (\underline{w}_1(x), \underline{w}_2(x))$ and $\overline{w} = (\overline{w}_1(x), \overline{w}_2(x)) \in X^2$.

We define the set $H \subset X^2$ by

$$H = \{(w_1, w_2) \in X^2 : 0 \le w_1 \le 1 \text{ and } 0 \le w_2 \le b - 1\}.$$

Our initial datum will always be chosen in the set of H. Here, we point out that the set H is positively invariant under the semiflow $\{S(t)\}_{t\ge 0}$ generated by system (1.1). In particular, this means that system (1.1) with initial condition (1.2) admits a unique globally defined solution $(U(x,t),V(x,t))$ with

$$(U,V)(x,\cdot)\in C^1([0,\infty),X^2),\quad \forall x\in\mathbb{R}\quad \text{and}\quad (U,V)(\cdot,t)\in H,\quad \forall t\geq 0.$$

Based on the research of Choi *et al.* [7], we obtain the spreading speed of the population of the prey by taking $\alpha\equiv 1$ and $v\equiv 0$ in the u-equation of (1.1), which is given by the quantity

$$s^*:=\inf_{0<\lambda<+\infty}\frac{d_1\left[\int_{\mathbb{R}}J_1(y)\mathrm{e}^{\lambda y}\mathrm{d}y-1\right]+r_1}{\lambda}. \tag{2.1}$$

Since $b>1$, we obtain the spreading speed of the predator population when the density of the prey is fixed to its maximal capacity 1, namely,

$$s_*:=\inf_{0<\lambda<+\infty}\frac{d_2\left[\int_{\mathbb{R}}J_2(y)\mathrm{e}^{\lambda y}\mathrm{d}y-1\right]+r_2(b-1)}{\lambda}. \tag{2.2}$$

Letting $s>0$ be a given fixed constant, we assume that

$$\underline{s}:=\min\{s^*,s_*\}>s.$$

Next, we give the following proposition for the spreading speed of a nonlocal system which will be used in Sections 3 and 4:

Proposition 2.1 ([17]) Let w be a solution of the system

$$\begin{cases}\dfrac{\partial w}{\partial t}(x,t)=d(J*w-w)+rw(x,t)(k-w(x,t)), & x\in\mathbb{R},t>0,\\ w(x,0)=\chi(x), & x\in\mathbb{R},\end{cases} \tag{2.3}$$

where kernel J satisfies (J1)–(J2) and the initial data $\chi\in X_s$ admits a nonempty compact support. The parameters satisfy $d>0,r>0$ and $k>0$. Let $w(\cdot,t)\in X_s$ for all $t>0$ for a given $\chi\in X_s$ and $\bar{c}:=\inf\limits_{0<\lambda<+\infty}\frac{d\left[\int_{\mathbb{R}}J(x)\mathrm{e}^{\lambda x}\mathrm{d}x-1\right]+rk}{\lambda}>0$. Then the following statements are valid:

(i) for any $c>\bar{c}$, if χ has a nonempty compact support, then

$$\lim_{t\to\infty}\sup_{|x|>ct}w(x,t)=0;$$

(ii) for any $0<c<\bar{c}$, if $\chi(\cdot)\not\equiv 0$, then

$$\liminf_{t\to\infty}\inf_{|x|<ct}w(x,t)=k.$$

We give the following lemma, which plays an important role in performing various limiting arguments:

Lemma 2.2 Let $\bar{\alpha}=\max\{-\alpha(-\infty),1\}$. For any $c\in(s,\underline{s})$, we assume that (J1), (J2), (α1)–(α3), (H1) and that $d_1>r_1\bar{\alpha}+\frac{r_1a}{2}+\frac{r_2b(b-1)}{2}$ and $d_2>r_2(b-1)+\frac{r_2b(b-1)}{2}+\frac{ar_1}{2}$. For any initial data satisfying $(u_0,v_0)\in H$ with $u_0\not\equiv 0$ and $v_0\not\equiv 0$, the corresponding solution (u,v) of (1.1) satisfies that

$$(u,v)(x+ct_n,t+t_n)\to(u_\infty,v_\infty)(x,t)\ \text{locally uniformly}\ \text{ as } n\to\infty,$$

where $\{t_n\}_{n\in\mathbb{Z}}$ is such that $t_n\to\infty$, as $n\to\infty$ and $(u_\infty,v_\infty)(x,t)$ satisfies that

$$\begin{cases}\dfrac{\partial u_\infty}{\partial t}(x,t)=d_1\left[\left(J_1*u_\infty\right)(x,t)-u_\infty(x,t)\right]+r_1u_\infty(x,t)\left[1-u_\infty(x,t)-av_\infty(x,t)\right],\\ \dfrac{\partial v_\infty}{\partial t}(x,t)=d_2\left[\left(J_2*v_\infty\right)(x,t)-v_\infty(x,t)\right]+r_2v_\infty(x,t)\left[-1+bu_\infty(x,t)-v_\infty(x,t)\right].\end{cases}$$

Proof Let $\{t_n\}_{n\in\mathbb{Z}}$ be such that $t_n \to \infty$, as $n \to \infty$. Define that

$$\begin{cases} u_n(x,t) = u\left(x + ct_n, t + t_n\right), \\ v_n(x,t) = v\left(x + ct_n, t + t_n\right), \\ \alpha_n(x - st) = \alpha(x - st + (c-s)t_n) \end{cases}$$

for $(x,t) \in \mathbb{R} \times [-t_n, +\infty)$. It is clear that $(u_n(x,t), v_n(x,t))$ satisfies that

$$\begin{cases} \dfrac{\partial u_n}{\partial t}(x,t) = d_1\left[\left(J_1 * u_n\right)(x,t) - u_n(x,t)\right] + r_1 u_n(x,t)\left[\alpha_n(x-st) - u_n(x,t) - av_n(x,t)\right], \\ \dfrac{\partial v_n}{\partial t}(x,t) = d_2\left[\left(J_2 * v_n\right)(x,t) - v_n(x,t)\right] + r_2 v_n(x,t)\left[-1 + bu_n(x,t) - v_n(x,t)\right], \\ u_n\left(x, -t_n\right) = u\left(x + ct_n, 0\right), v_n\left(x, -t_n\right) = v\left(x + ct_n, 0\right). \end{cases}$$

Next, we use some *a prior* estimates of $(u_n(x,t), v_n(x,t))$ uniformly in n to reach to the limit as $n \to +\infty$. Since

$$0 \le u(x,0) = u_0 \le 1, 0 \le v(x,0) = v_0 \le b-1,$$

we have that

$$0 \le u_n(x, -t_n) \le 1, 0 \le v_n(x, -t_n) \le b-1.$$

Hence, $0 \le u_n(x,t) \le 1, 0 \le v_n(x,t) \le b-1$. By assumptions (α_1) and (α_2), we have that

$$|\alpha_n(x-st)| \le \max\{-\alpha(-\infty), 1\} = \bar{\alpha}. \tag{2.4}$$

Therefore, there are positive constants $D_i, i = 1, 2, \cdots, 4$, such that, for $(x,t) \in \mathbb{R} \times [-t_n, +\infty)$ and $n \in \mathbb{Z}$,

$$\begin{aligned} \left|\frac{\partial u_n}{\partial t}\right| &\le d_1\left|J_1 * u_n\right| + d_1\left|u_n\right| + r_1\left|u_n\right|\left[\left|\alpha_n(x-st)\right| + \left|u_n\right| + a\left|v_n\right|\right] \\ &\le 2d_1 + r_1(1 + \bar{\alpha} + a(b-1)) =: D_1, \\ \left|\frac{\partial v_n}{\partial t}\right| &\le d_2\left|J_2 * v_n\right| + d_2\left|v_n\right| + r_2\left|v_n\right|\left[1 + b\left|u_n\right| + \left|v_n\right|\right] \le (b-1)\left(2d_2 + 2br_2\right) =: D_2. \end{aligned}$$

It follows from assumption (α_3) that there exists a constant $M > 0$ such that, for any $n \in \mathbb{N}$,

$$|(\alpha_n)_t| \le M, \text{ for any } t \in \mathbb{R}. \tag{2.5}$$

Therefore, by using (2.5) and the assumption $c \in (s, \underline{s})$, we have that

$$\begin{aligned} \left|\frac{\partial^2 u_n}{\partial t^2}\right| &\le d_1|J_1 * (u_n)_t| + d_1|(u_n)_t| + r_1|(u_n)_t|\left[|\alpha_n(x-st)| + |u_n| + a|v_n|\right] \\ &\quad + r_1\left|u_n\right|\left[\left|-s*(\alpha_n)_t\right| + \left|\left(u_n\right)_t\right|\right) + a\left|(v_n)_t\right|\right] \\ &\le 2d_1D_1 + r_1D_1(1 + \bar{\alpha} + ab - a) + r_1(\underline{s}M + D_1 + aD_2) =: D_3, \\ \left|\frac{\partial^2 v_n}{\partial t^2}\right| &\le 2d_2D_2 + 2br_2D_2 + r_2(b-1)(bD_1 + D_2) =: D_4. \end{aligned}$$

For any $\gamma > 0$, define that

$$\begin{cases} \mathcal{U}_{n,\gamma}(x,t) := u_n(x+\gamma, t) - u_n(x,t), \\ \mathcal{V}_{n,\gamma}(x,t) := v_n(x+\gamma, t) - v_n(x,t), \\ \tilde{J}_i(x) := J_i(x+\gamma) - J_i(x), i = 1, 2. \end{cases}$$

Since J_i satisfies (J1) and (J2), $J_i' \in L^1$, and there exists $L_i > 0, i = 1, 2$ such that

$$\begin{aligned}\int_{\mathbb{R}} \left|\tilde{J}_i(x-y)\right| \mathrm{d}y &= \int_{\mathbb{R}} |J_i(x+\gamma-y) - J_i(x-y)| \,\mathrm{d}y \\ &= |\gamma| \int_{\mathbb{R}} \left| \int_0^1 J_i'(x-y+\theta\gamma)\mathrm{d}\theta \right| \mathrm{d}y \\ &\le |\gamma| \int_0^1 \int_{\mathbb{R}} |J_i'(x-y+\theta\gamma)| \,\mathrm{d}y\mathrm{d}\theta \le L_i|\gamma|.\end{aligned}$$

By assumptions (α_1) and (α_3), there exists $L_3 > 0$ such that

$$|\alpha_n(x+\gamma-st) - \alpha_n(x-st)| \le L_3|\gamma|.$$

Hence, for any $\eta > 0$, there exists $\delta_i = \frac{\eta}{L_i} > 0$ $(i = 1, 2, 3)$ such that

$$\int_{\mathbb{R}} |\tilde{J}_i(x-y)|\mathrm{d}y \le \eta \text{ and } |\alpha_n(x+\gamma-st) - \alpha_n(x-st)| \le \eta,$$

provided that $|\gamma| \le \delta_i, x \in \mathbb{R}, i = 1, 2, 3$. Using (2.4), we can verify that

$$\begin{aligned}&\frac{\partial}{\partial t}\mathcal{U}_{n,\gamma}^2(x,t) = 2\mathcal{U}_{n,\gamma}(x,t)\frac{\partial \mathcal{U}_{n,\gamma}}{\partial t}(x,t) \\ &= 2\mathcal{U}_{n,\gamma}(x,t)\Big(d_1\int_{\mathbb{R}} \tilde{J}_1(x-y)u_n(y,t)\mathrm{d}y - (d_1 - r_1\alpha_n(x+\gamma-st))\mathcal{U}_{n,\gamma}(x,t) \\ &\quad - r_1\mathcal{U}_{n,\gamma}(x,t)(u_n(x+\gamma,t)+u_n(x,t)) + r_1u_n(x,t)(\alpha_n(x+\gamma-st)-\alpha_n(x-st)) \\ &\quad - r_1av_n(x+\gamma,t)\mathcal{U}_{n,\gamma}(x,t) - r_1au_n(x,t)\mathcal{V}_{n,\gamma}(x,t)\Big) \\ &\le 2\mathcal{U}_{n,\gamma}(x,t)\Big(d_1\int_{\mathbb{R}} \tilde{J}_1(x-y)u_n(y,t)\mathrm{d}y - (d_1 - r_1\alpha_n(x+\gamma-st))\mathcal{U}_{n,\gamma}(x,t) \\ &\quad - r_1\mathcal{U}_{n,\gamma}(x,t)(u_n(x+\gamma,t)+u_n(x,t)) + r_1u_n(x,t)(\alpha_n(x+\gamma-st)-\alpha_n(x-st))\Big) \\ &\quad + r_1au_n(x,t)\left(\mathcal{U}_{n,\gamma}^2(x,t) + \mathcal{V}_{n,\gamma}^2(x,t)\right) \\ &\le 4(d_1+r_1)\eta - 2\left(d_1 - r_1\bar{\alpha} - \frac{ar_1}{2}\right)\mathcal{U}_{n,\gamma}^2(x,t) + ar_1\mathcal{V}_{n,\gamma}^2(x,t), \end{aligned} \tag{2.6}$$

and

$$\begin{aligned}&\frac{\partial}{\partial t}\mathcal{V}_{n,\gamma}^2(x,t) = 2\mathcal{V}_{n,\gamma}(x,t)\frac{\partial \mathcal{V}_{n,\gamma}}{\partial t}(x,t) \\ &= 2\mathcal{V}_{n,\gamma}(x,t)\Big(d_2\int_{\mathbb{R}} \tilde{J}_2(x-y)v_n(y,t)\mathrm{d}y - (d_2+r_2)\mathcal{V}_{n,\gamma}(x,t) \\ &\quad - r_2(v_n(x+\gamma,t)+v_n(x,t))\mathcal{V}_{n,\gamma}(x,t) + r_2bu_n(x,t)\mathcal{V}_{n,\gamma}(x,t) + r_2bv_n(x+\gamma,t)\mathcal{U}_{n,\gamma}(x,t)\Big) \\ &\le 2\mathcal{V}_{n,\gamma}(x,t)\Big(d_2\int_{\mathbb{R}} \tilde{J}_2(x-y)v_n(y,t)\mathrm{d}y - (d_2+r_2)\mathcal{V}_{n,\gamma}(x,t) + r_2bu_n(x,t)\mathcal{V}_{n,\gamma}(x,t)\Big) \\ &\quad + r_2bv_n(x+\gamma,t)(\mathcal{U}_{n,\gamma}^2(x,t) + \mathcal{V}_{n,\gamma}^2(x,t)) \\ &\le 4d_2\eta(b-1) - 2\left(d_2 + r_2 - r_2b - \frac{r_2b(b-1)}{2}\right)\mathcal{V}_{n,\gamma}^2(x,t) + r_2b(b-1)\mathcal{U}_{n,\gamma}^2(x,t). \end{aligned} \tag{2.7}$$

Adding the two inequalities (2.6) and (2.7), we deduce from the assumptions that

$$k_1 := d_1 - r_1\bar{\alpha} - \frac{r_1a}{2} - \frac{r_2b(b-1)}{2} > 0 \quad\text{and}\quad k_2 := d_2 + r_2 - r_2b - \frac{r_2b(b-1)}{2} - \frac{ar_1}{2} > 0.$$

Then we obtain that

$$\frac{\partial}{\partial t}(\mathcal{U}_{n,\gamma}^2(x,t) + \mathcal{V}_{n,\gamma}^2(x,t))$$

$$\begin{aligned}&\le 4\left(d_1+r_1+d_2(b-1)\right)\eta-2\left(d_1-r_1\bar{\alpha}-\frac{r_1 a}{2}-\frac{r_2 b(b-1)}{2}\right)\mathcal{U}_{n,\gamma}^2(x,t)\\&\quad-2\left(d_2+r_2-r_2 b-\frac{r_2 b(b-1)}{2}-\frac{a r_1}{2}\right)\mathcal{V}_{n,\gamma}^2(x,t)\\&=4\left(d_1+r_1+d_2(b-1)\right)\eta-2k_1\mathcal{U}_{n,\gamma}^2(x,t)-2k_2\mathcal{V}_{n,\gamma}^2(x,t).\end{aligned}\tag{2.8}$$

Let $k=\min\{k_1,k_2\}$. Due to (2.8), we have that

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\mathcal{U}_{n,\gamma}^2(x,t)+\mathcal{V}_{n,\gamma}^2(x,t)\right)\\&\le 4\left(d_1+r_1+d_2(b-1)\right)\eta-2k\left(\mathcal{U}_{n,\gamma}^2(x,t)+\mathcal{V}_{n,\gamma}^2(x,t)\right).\end{aligned}\tag{2.9}$$

Multiplying both sides of (2.9) by $\mathrm{e}^{2k(t-s)}$ and integrating from s to t, we have that

$$\begin{aligned}&\left(\mathcal{U}_{n,\gamma}^2(x,t)+\mathcal{V}_{n,\gamma}^2(x,t)\right)\\&\le \mathrm{e}^{-2k(t-s)}\left(\mathcal{U}_{n,\gamma}^2(x,s)+\mathcal{V}_{n,\gamma}^2(x,s)\right)+4\left(d_1+r_1+d_2(b-1)\right)\eta\int_s^t \mathrm{e}^{-2k(t-\theta)}\mathrm{d}\theta.\end{aligned}\tag{2.10}$$

Taking $s=-t_n$ from (2.10), we get that

$$\begin{aligned}&\left(\mathcal{U}_{n,\gamma}^2(x,t)+\mathcal{V}_{n,\gamma}^2(x,t)\right)\\&\le \mathrm{e}^{-2k(t+t_n)}\left(\mathcal{U}_{n,\gamma}^2\left(x,-t_n\right)+\mathcal{V}_{n,\gamma}^2\left(x,-t_n\right)\right)+\frac{2\left(d_1+r_1+d_2(b-1)\right)\eta}{k};\end{aligned}$$

that is,

$$\begin{aligned}&\left|u_n\left(x+\gamma,t\right)-u_n\left(x,t\right)\right|^2+\left|v_n\left(x+\gamma,t\right)-v_n\left(x,t\right)\right|^2\\&\le\left|u_n(x+\gamma,-t_n)-u_n(x,-t_n)\right|^2+\left|v_n(x+\gamma,-t_n)-v_n(x,-t_n)\right|^2+\frac{2\left(d_1+r_1+d_2(b-1)\right)\eta}{k}.\end{aligned}$$

Since $u_n(x,-t_n)$ and $v_n(x,-t_n)$ are uniformly continuous for $x\in\mathbb{R}$, there exists $\delta_4>0$ such that $|u_n(x+\gamma,-t_n)-u_n(x,-t_n)|\le\eta^{1/2}$ and $|v_n(x+\gamma,-t_n)-v_n(x,-t_n)|\le\eta^{1/2}$, whatever we have for $|\gamma|\le\delta_4$. Thus, there exists a positive constant D_5, and, for any $\gamma>0$, such that $|\gamma|\le\delta:=\min\{\delta_1,\delta_2,\delta_3,\delta_4\}$, and we have that, for all $x\in\mathbb{R}$ and $t>-t_n$,

$$\begin{cases}|u_n(x+\gamma,t)-u_n(x,t)|^2\le\left(2+\dfrac{2\left(d_1+r_1+d_2(b-1)\right)}{k}\right)\eta:=D_5\eta,\\|v_n(x+\gamma,t)-v_n(x,t)|^2\le\left(2+\dfrac{2\left(d_1+r_1+d_2(b-1)\right)}{k}\right)\eta:=D_5\eta.\end{cases}$$

Furthermore, there exist positive constants D_6 and D_7 such that, for all $x\in\mathbb{R}$ and $t>-t_n$, it follows that

$$\begin{aligned}&\left|\frac{\partial u_n}{\partial t}(x+\gamma,t)-\frac{\partial u_n}{\partial t}(x,t)\right|\\&\le\Big|d_1\left(J_1*\left(u_n(x+\gamma,t)-u_n(x,t)\right)\right)-d_1\left(u_n(x+\gamma,t)-u_n(x,t)\right)\\&\quad-r_1\left(u_n(x+\gamma,t)+u_n(x,t)\right)\left(u_n(x+\gamma,t)-u_n(x,t)\right)\\&\quad-ar_1v_n(x+\gamma,t)\left(u_n(x+\gamma,t)-u_n(x,t)\right)-ar_1u_n(x,t)\left(v_n(x+\gamma,t)-v_n(x,t)\right)\\&\quad+\alpha_n(x+\gamma-st)r_1\left(u_n(x+\gamma,t)-u_n(x,t)\right)+r_1(\alpha_n(x+\gamma-st)-\alpha_n(x-st))u_n(x,t)\Big|\\&\le(2d_1+2r_1+ar_1+\bar{\alpha}r_1+ar_1b)D_5\eta+r_1\eta=:D_6\eta,\end{aligned}$$

and

$$\left|\frac{\partial v_n}{\partial t}(x+\gamma,t)-\frac{\partial v_n}{\partial t}(x,t)\right|\le D_7\eta.$$

Since $c\in(s,\underline{s})$ and on account of the assumption (α2), we have that

$$\lim_{n\to\infty}\alpha_n(x-st)=\lim_{n\to\infty}\alpha(x-st+(c-s)t_n)=1$$

locally uniformly with respect to $x\in\mathbb{R}$ and $t\in\mathbb{R}$. By the above *a prior* estimates, the Arzelà-Ascoli theorem and a diagonal extraction process, we can extract a subsequence $t_n\to\infty$ such that

$$(u_n,v_n)(x,t)\to(u_\infty,v_\infty)(x,t) \text{ locally uniformly as } n\to\infty,$$

where $(u_\infty,v_\infty)(x,t)$ satisfies that

$$\begin{cases}\dfrac{\partial u_\infty}{\partial t}(x,t)=d_1\left[(J_1*u_\infty)(x,t)-u_\infty(x,t)\right]+r_1u_\infty(x,t)\left[1-u_\infty(x,t)-av_\infty(x,t)\right],\\ \dfrac{\partial v_\infty}{\partial t}(x,t)=d_2\left[(J_2*v_\infty)(x,t)-v_\infty(x,t)\right]+r_2v_\infty(x,t)\left[-1+bu_\infty(x,t)-v_\infty(x,t)\right].\end{cases}$$

We complete the proof of Lemma 2.2. □

3 Survival of the Prey u

In this section, we consider the large time behavior of solutions of system (1.1) with an initial value (1.2), and more precisely we deal with the persistence of the prey u in the moving frames with speeds between s and $\underline{s}=\min\{s^*,s_*\}>s$.

Theorem 3.1 (Uniform spreading of u) Let $\bar\alpha=\max\{-\alpha(-\infty),1\}$. Assume that (J1), (J2), (α1)–(α3), (H1) and $d_1>r_1\bar\alpha+\frac{r_1a}{2}+\frac{r_2b(b-1)}{2}$ and $d_2>r_2(b-1)+\frac{r_2b(b-1)}{2}+\frac{ar_1}{2}$. Let the initial data $(u_0,v_0)\in H$ with $u_0\not\equiv0$ and $v_0\not\equiv0$ be given. If $s<\underline{s}$, then, for any $\eta\in(0,(\underline{s}-s)/2)$, there exists $\varepsilon>0$ such that $\liminf\limits_{t\to+\infty}\inf\limits_{(s+\eta)t\le|x|\le(\underline{s}-\eta)t}u(x,t)\ge\varepsilon$.

We divide things into three steps to prove Theorem 3.1. We first use Lemma 3.2 to prove the "pointwise weak spreading", which illustrates that the u-component of the solution of system (1.1) does not converge to 0. Then we apply Lemma 3.4 to show the "pointwise spreading", which means that the solution is bounded along with the path $x=ct$ by some constant $\varepsilon>0$ as $t\to+\infty$. We complete the proof of Theorem 3.1 by showing that the spreading is in fact uniform in the intermediate range between the moving frames with speeds s and $\underline{s}=\min\{s_*,s^*\}$.

Lemma 3.2 (Pointwise weak spreading) Assume that $s<\underline{s}$. Then, for any $c\in(s,\underline{s})$, there exists $\varepsilon_1(c)>0$ such that, for any $(u_0,v_0)\in H$ with $u_0\not\equiv0$ and $v_0\not\equiv0$, the corresponding solution (u,v) of (1.1) satisfies that $\limsup\limits_{t\to\infty}u(ct,t)\ge\varepsilon_1(c)$.

Proof We argue by contradiction by assuming that there exists a sequence

$$\{(u_{0,n},v_{0,n})\}_{n\ge0}\in H,$$

such that $u_{0,n},v_{0,n}\not\equiv0$ and

$$\lim_{n\to+\infty}\limsup_{t\to+\infty}u_n(ct,t)=0, \tag{3.1}$$

where (u_n,v_n) is the solution of system (1.1) with an initial value $(u_{0,n},v_{0,n})$. Then we can choose a time sequence $t_n\to+\infty$ as $n\to\infty$ such that $\lim\limits_{n\to+\infty}\sup\limits_{t\ge t_n}u_n(ct,t)=0$.

Now we claim that, for any $R>0$,

$$\lim_{n\to+\infty}\sup_{|x|\le R, t\ge t_n} u_n(x+ct,t)=0. \tag{3.2}$$

Indeed, assume, by contradiction, that there exist sequences $x_n\in[-R,R]$ and $t'_n\ge t_n$ such that

$$\liminf_{n\to+\infty} u_n(x_n+ct'_n,t'_n)>0. \tag{3.3}$$

By arguments similar to those for Lemma 2.2 and $c>s$, we can extract a subsequence such that the following convergence holds locally uniform in $(x,t)\in\mathbb{R}\times\mathbb{R}$:

$$\begin{cases}\lim\limits_{n\to\infty} u_n\left(x+ct'_n,t+t'_n\right)=u_\infty(x,t),\\ \lim\limits_{n\to\infty} v_n\left(x+ct'_n,t+t'_n\right)=v_\infty(x,t).\end{cases}$$

Here the limit function (u_∞,v_∞) is an entire solution of the system

$$\begin{cases}\dfrac{\partial u_\infty}{\partial t}(x,t)=d_1\left[\left(J_1*u_\infty\right)(x,t)-u_\infty(x,t)\right]+r_1u_\infty(x,t)\left[1-u_\infty(x,t)-av_\infty(x,t)\right],\\ \dfrac{\partial v_\infty}{\partial t}(x,t)=d_2\left[\left(J_2*v_\infty\right)(x,t)-v_\infty(x,t)\right]+r_2v_\infty(x,t)\left[-1+bu_\infty(x,t)-v_\infty(x,t)\right].\end{cases} \tag{3.4}$$

It is easy to see that $u_\infty\ge 0$, and we deduce from (3.1) that $u_\infty(0,0)=0$. According to the strong maximum principle, we obtain that $u_\infty\equiv 0$. On the other hand, by (3.3), we can also extract another subsequence, $x_n\to x_\infty\in[-R,R]$, such that $u_\infty(x_\infty,0)>0$, which is a contradiction. Therefore, (3.2) holds.

Similarly, we claim that

$$\lim_{n\to+\infty}\sup_{|x|\le R, t\ge t_n} v_n(x+ct,t)=0. \tag{3.5}$$

Indeed, if this is not true, then we can find an entire in time solution (u_∞,v_∞) of (3.4) with $u_\infty\equiv 0$ and $v_\infty>0$. Then

$$\begin{aligned}\frac{\partial v_\infty}{\partial t}(x,t)&=d_2\left[\left(J_2*v_\infty\right)(x,t)-v_\infty(x,t)\right]+r_2v_\infty(x,t)\left(-1-v_\infty(x,t)\right)\\ &\le d_2\left[\left(J_2*v_\infty\right)(x,t)-v_\infty(x,t)\right]-r_2v_\infty(x,t)\end{aligned} \tag{3.6}$$

for all $x\in\mathbb{R}$ and $t\in\mathbb{R}$. Let $\bar v_\infty(x,t)$ be the solution of

$$\begin{cases}\dfrac{\partial \bar v_\infty}{\partial t}(x,t)=d_2\left[\left(J_2*\bar v_\infty\right)(x,t)-\bar v_\infty(x,t)\right]-r_2\bar v_\infty(x,t), x\in\mathbb{R}, t>-t_0, t_0\in\mathbb{R},\\ \bar v_\infty(x,-t_0)=v_\infty(x,-t_0), x\in\mathbb{R}, t_0\in\mathbb{R}.\end{cases} \tag{3.7}$$

It is easy to verify that, for any $t_0\in\mathbb{R}$, the function $(x,t)\mapsto(b-1)\mathrm{e}^{-r_2(t+t_0)}$ is an upper solution of the above equation for any $t>-t_0$. By using (3.6) and the comparison principle, we have that

$$v_\infty(x,t)\le(b-1)\mathrm{e}^{-r_2(t+t_0)}, x\in\mathbb{R}, t>-t_0. \tag{3.8}$$

Due to $v_\infty(x,-t_0)\le b-1$ for any $t_0\in\mathbb{R}^+$ and (3.8), we deduce that $v_\infty(x,0)\le(b-1)\mathrm{e}^{-r_2t_0}$. Then we have that $v_\infty(x,0)\equiv 0$ as $t_0\to\infty$. By the strong maximum principle, we obtain that $v_\infty\equiv 0$. This contradicts $v_\infty>0$. The claim (3.5) is now proven.

By (3.5), for any $\delta_1>0$, there exists n large enough such that

$$v_n(x,t)\le\delta_1,\ \text{for all } (x,t) \text{ such that } t\ge t_n, x\in(ct-R,ct+R). \tag{3.9}$$

Since $\alpha(\infty)=1$, and by (3.9), there exists n large enough such that

$$\frac{\partial u_n}{\partial t}(x,t)\geq d_1\left[\left(J_1*u_n\right)(x,t)-u_n(x,t)\right]+r_1(1-u_n(x,t)-a\delta_1)u_n(x,t) \tag{3.10}$$

for all $t\geq t_n$ and $x\in(ct-R,ct+R)$. On the other hand, set

$$Q[W](x,t):=d_1\int_{\mathbb{R}}J_1(x-y)W(y,t)\mathrm{d}y-d_1W(x,t)+r_1W(x,t)(1-W(x,t)-a\delta_1)-\partial_tW(x,t), \tag{3.11}$$

and

$$L[W](x,t):=d_1\int_{\mathbb{R}}J_1(x-y)W(y,t)\mathrm{d}y+mW(x,t)-\partial_tW(x,t), \tag{3.12}$$

where m will be determined later. Let

$$\phi(x,t):=\phi_{R,\beta,\eta}(x,t)=\begin{cases}\eta\mathrm{e}^{r_1a\delta_1t}\mathrm{e}^{-\beta(x-ct)}\cos\left(\dfrac{\pi(x-ct)}{2R}\right), & x\in(-R+ct,R+ct),t\in\mathbb{R},\\ 0, & x\in\mathbb{R}\setminus(-R+ct,R+ct),t\in\mathbb{R},\end{cases} \tag{3.13}$$

where $\eta\in(0,+\infty)$, $\beta>0$, and we assume that β and η are two independent constants. Next we show that $\phi(x,t)$ is a sub-solution of (3.11). In fact, we first claim that, for R large enough,

$$\begin{aligned}J_1*\phi(x,t)&=\eta\int_{-R+ct}^{R+ct}J_1(x-y)\mathrm{e}^{r_1a\delta_1t}\mathrm{e}^{-\beta(y-ct)}\cos\left(\frac{\pi(y-ct)}{2R}\right)\mathrm{d}y\\&=\eta\int_{-R}^{R}J_1(x-ct-y)\mathrm{e}^{r_1a\delta_1t}\mathrm{e}^{-\beta y}\cos\left(\frac{\pi y}{2R}\right)\mathrm{d}y\\&\geq\eta\int_{-\infty}^{\infty}J_1(x-ct-y)\mathrm{e}^{r_1a\delta_1t}\mathrm{e}^{-\beta y}\cos\left(\frac{\pi y}{2R}\right)\mathrm{d}y.\end{aligned} \tag{3.14}$$

Indeed, due to the fact that are anyhow have that $J_1*\phi(x,t)\geq0$, we take, without loss of generality, that $x\in(-R+ct,R+ct)$ and $t\in\mathbb{R}$. In order to make (3.14) valid we have to show that, for $y\in\mathbb{R}\setminus(-R+ct,R+ct)$, either $\cos\left(\frac{\pi y}{2R}\right)\leq0$ or $J_1(x-ct-y)=0$. We assume that J_1 has a compact support, so there exists K such that supp $J_1\subset[-K,K]$. If $x\in(-R+ct,R+ct)$ and $|x-ct-y|\leq K$, then $y\in(-R-K,R+K)\subset(-3R,3R)$ when $K\leq2R$. Thus, we obtain that $\cos\left(\frac{\pi y}{2R}\right)\leq0$ for $y\in[-3R,-R]\cup[R,3R]$. Moreover, since supp $J_1\subset[-K,K]\subset(-2R,2R)$ when $K<2R$, we obtain that $J_1(x-ct-y)=0$ for $y\in\mathbb{R}\setminus(-3R,3R)$. Thus, we deduce that

$$\begin{aligned}&\eta\int_{-\infty}^{\infty}J_1(x-ct-y)\mathrm{e}^{r_1a\delta_1t}\mathrm{e}^{-\beta y}\cos\left(\frac{\pi y}{2R}\right)\mathrm{d}y\\&=\left(\int_{-\infty}^{-3R}+\int_{-3R}^{-R}+\int_{-R}^{R}+\int_{R}^{3R}+\int_{3R}^{\infty}\right)J_1(x-ct-y)\eta\mathrm{e}^{r_1a\delta_1t}\mathrm{e}^{-\beta y}\cos\left(\frac{\pi y}{2R}\right)\mathrm{d}y\\&\leq\eta\int_{-R}^{R}J_1(x-ct-y)\mathrm{e}^{r_1a\delta_1t}\mathrm{e}^{-\beta y}\cos\left(\frac{\pi y}{2R}\right)\mathrm{d}y;\end{aligned}$$

that is, (3.14) holds. Taking (3.13) into (3.12) and using (3.14), we obtain that

$$\begin{aligned}L[\phi](x,t)\geq{}&c\left[-\beta\phi-\frac{\pi\eta}{2R}\mathrm{e}^{-\beta(x-ct)}\sin\left(\frac{\pi(x-ct)}{2R}\right)\mathrm{e}^{r_1a\delta_1t}\right]+(m-r_1a\delta_1)\phi\\&+d_1\eta\int_{-\infty}^{\infty}J_1(x-ct-y)\mathrm{e}^{-\beta y}\cos\left(\frac{\pi y}{2R}\right)\mathrm{e}^{r_1a\delta_1t}\mathrm{d}y\\={}&\left[-c\beta+m-r_1a\delta_1+d_1\int_{\mathbb{R}}\mathrm{e}^{\beta y}J_1(y)\cos\left(\frac{\pi y}{2R}\right)\mathrm{d}y\right]\phi\end{aligned}$$

$$+\left[-\frac{\pi}{2R}c+d_1\int_{\mathbb{R}}\mathrm{e}^{\beta y}J_1(y)\sin\left(\frac{\pi y}{2R}\right)\mathrm{d}y\right]\eta\mathrm{e}^{-\beta(x-ct)}\mathrm{e}^{r_1a\delta_1t}\sin\left(\frac{\pi(x-ct)}{2R}\right).$$

Therefore, $L[\phi]>0$ on $x\in[-R+ct,R+ct]$ and $t\in\mathbb{R}$ if the following two conditions are satisfied:

$$c<\frac{1}{\beta}\left[m-r_1a\delta_1+d_1\int_{\mathbb{R}}\mathrm{e}^{\beta y}J_1(y)\cos\left(\frac{\pi y}{2R}\right)\mathrm{d}y\right]=:\mathcal{A}_m(\beta,R);\tag{3.15}$$

$$c=\frac{2Rd_1}{\pi}\left[\int_{\mathbb{R}}\mathrm{e}^{\beta y}J_1(y)\sin\left(\frac{\pi y}{2R}\right)\mathrm{d}y\right]=:\mathcal{B}(\beta,R).\tag{3.16}$$

We first establish some properties of the functions $\mathcal{A}_m$ and $\mathcal{B}$. As $R\to\infty$, we have the locally uniform convergence of

$$\mathcal{A}_m(\beta,R)\to A_m(\beta)=\frac{m-r_1a\delta_1+d_1\int_{\mathbb{R}}\mathrm{e}^{\beta y}J_1(y)\mathrm{d}y}{\beta},$$

$$\mathcal{B}(\beta,R)\to B(\beta):=d_1\int_{\mathbb{R}}y\mathrm{e}^{\beta y}J_1(y)\mathrm{d}y.$$

Differentiation gives that

$$A_m'(\beta)=\left(B(\beta)-A_m(\beta)\right)/\beta,\quad B'(\beta)=d_1\int_{\mathbb{R}}J(y)\mathrm{e}^{\beta y}y^2\mathrm{d}y>0.\tag{3.17}$$

It follows from the properties of the function $A_m(\beta)$ that it achieves infimum. Then, there exists $\beta^*>0$ such that $A_m(\beta^*)=\inf_{\beta>0}A_m(\beta)$. By the definition of β^* and (3.17), we obtain that $B(\beta^*)=A_m(\beta^*)$. Since B is an increasing function, $B(\beta)<B(\beta^*)$ for $0<\beta<\beta^*$. Then we have that

$$A_m(\beta)>A_m(\beta^*)=B(\beta^*)>B(\beta),\qquad \text{for } 0<\beta<\beta^*.\tag{3.18}$$

In addition, we define that $c^*:=A_{m^*}(\beta^*)$ with $m^*=r_1-r_1\delta_2-r_1a\delta_1-d_1$ and $r_1-r_1\delta_2-2r_1a\delta_1>0$ for small enough constants $\delta_1>0$, $\delta_2>0$. Since $0<s<c<\underline{s}$, we can choose $m<m^*$ such that $0<s<c<A_m(\beta^*)<A_{m^*}(\beta^*)=c^*$. Noting that $B(0)<c^*$ and $B(0)=0$, we have that $c>s>B(0)=0$. Then, combing this with (3.18), we can choose $c_1,c_2,\delta,R>0$ such that

$$B(c_1)+\delta<c<B(c_2)-\delta \text{ and } |\mathcal{B}(\beta,R)-B(\beta)|<\delta.$$

It follows from the continuity of $\mathcal{B}(\beta,R)$ and $B(\beta)$ that there exists some $\beta(R)$ such that $\mathcal{B}(\beta(R),R)=c$ for all large enough R. Obviously, we can choose R large enough such that $\mathcal{A}_m(\beta(R),R)>c$. Thus, we have proven that (3.16) and (3.15) hold true.

By the definition of $\phi_{R,\beta,\eta}(x,t)$, we obtain that $L[\phi](x,t)>0$ for $(x,t)\in\mathbb{R}\times\mathbb{R}$. Note that $r_1W(1-W-a\delta_1)\geq(r_1-r_1\delta_2-r_1a\delta_1)W$ for $0\leq W\leq\delta_2$ and $m<r_1-r_1\delta_2-r_1a\delta_1-d_1$. Therefore, we have that

$$Q[\phi](x,t)>L[\phi](x,t)>0,\ \text{for } (x,t)\in\mathbb{R}\times\mathbb{R},$$

namely, that

$$-d_1\int_{\mathbb{R}}J_1(x-y)\phi(y,t)\mathrm{d}y-r_1\phi(x,t)(1-\phi(x,t)-a\delta_1)+\partial_t\phi(x,t)<0,\ \text{for } (x,t)\in\mathbb{R}\times\mathbb{R}.$$

Using (3.10) and taking η small enough so that $u_n(x,t_n)\geq\phi(x,t_n),\forall x\in\mathbb{R}$, we get, by the comparison principle, that $u_n(x,t)\geq\phi(x,t)$ for all $t\geq t_n$ and $x\in\mathbb{R}$. However, $\phi(ct,t)=\eta\mathrm{e}^{r_1a\delta_1t}\to+\infty$ as $t\to+\infty$, which contradicts $0\leq u_n\leq1$. This completes the proof of Lemma 3.2. $\square$

Lemma 3.3 For any $c \in [0, \underline{s})$, there exists $\varepsilon_1'(c) > 0$ such that, for any initial data satisfying $(u_0, v_0) \in H$ with $u_0 \not\equiv 0$ and $v_0 \not\equiv 0$, the solution (u, v) of the system

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = d_1\left[(J_1 * u)(x,t) - u(x,t)\right] + r_1 u(x,t)\left[1 - u(x,t) - av(x,t)\right], \\ \dfrac{\partial v}{\partial t}(x,t) = d_2\left[(J_2 * v)(x,t) - v(x,t)\right] + r_2 v(x,t)\left[-1 + bu(x,t) - v(x,t)\right] \end{cases} \tag{3.19}$$

satisfies that

$$\limsup_{t\to+\infty} u(ct,t) \geq \varepsilon_1'(c).$$

Proof The proof of Lemma 3.3 is similar to that of Lemma 3.2, so we omit it. □

Lemma 3.4 (Pointwise spreading) Assume that $s < \underline{s}$. Then, for any $(u_0, v_0) \in H$ with $u_0 \not\equiv 0$ and $v_0 \not\equiv 0$, for all $c \in (s, \underline{s})$, there exists $\varepsilon_2(c) > 0$ such that the corresponding solution (u, v) of (1.1) satisfies that

$$\liminf_{t\to+\infty} u(ct,t) \geq \varepsilon_2(c).$$

Proof We argue by contradiction to prove this assertion, i.e., that u spreads away from 0. We assume that there are sequences $(u_{0,n}, v_{0,n}) \in H$ with $u_{0,n} \not\equiv 0$ and $v_{0,n} \not\equiv 0$ such that

$$\lim_{n\to+\infty} u_n(ct_n, t_n) = 0. \tag{3.20}$$

By Lemma 3.2, there exists another sequence $t_n' \to \infty$ such that

$$\lim_{n\to+\infty} u_n(ct_n', t_n') \geq \frac{\varepsilon_1(c)}{2},$$

and without loss of generality, we can choose it so that $t_n' < t_n$ for any n. We define that

$$\tau_n := \sup\left\{t_n' \leq t \leq t_n | u_n(ct,t) \geq \frac{\varepsilon_1(c)}{2}\right\},$$

from which it follows that

$$\forall t \in (\tau_n, t_n), u_n(ct,t) \leq \frac{\varepsilon_1(c)}{2}. \tag{3.21}$$

Then this yields the following properties:

$$\begin{aligned} &u_n(c\tau_n, \tau_n) = \frac{\varepsilon_1(c)}{2}, \\ &u_n(ct,t) \leq \frac{\varepsilon_1(c)}{2}, \quad t \in (\tau_n, t_n), \\ &u_n(c(\tau_n + t_n - \tau_n), \tau_n + t_n - \tau_n) \to 0, \text{ as } n \to \infty. \end{aligned} \tag{3.22}$$

By Lemma 2.2, we can extract a subsequence $t_n \to \infty$ such that

$$(u_n, v_n)(x + ct_n, t + t_n) \to (u_\infty, v_\infty)(x,t) \text{ locally uniformly as } n \to \infty,$$

where $(u_\infty, v_\infty)(x,t)$ satisfies that

$$\begin{cases} \dfrac{\partial u_\infty}{\partial t}(x,t) = d_1\left[(J_1 * u_\infty)(x,t) - u_\infty(x,t)\right] + r_1 u_\infty(x,t)\left[1 - u_\infty(x,t) - av_\infty(x,t)\right], \\ \dfrac{\partial v_\infty}{\partial t}(x,t) = d_2\left[(J_2 * v_\infty)(x,t) - v_\infty(x,t)\right] + r_2 v_\infty(x,t)\left[-1 + bu_\infty(x,t) - v_\infty(x,t)\right]. \end{cases}$$

From the choice of t_n, we have that $u_\infty(0,0) = 0$, and hence that $u_\infty \equiv 0$, by the strong maximum principle. In particular, the sequence $t_n - \tau_n$ is unbounded. Indeed, assuming by contraction that $\lim_{n\to\infty}(t_n-\tau_n)=l<+\infty$, it follows from the third formula of (3.22) that

$$\lim_{n\to+\infty} u_n(c(\tau_n+t),\tau_n+t)=0,\quad \forall t\in[0,t_n-\tau_n],$$

which contradicts the fact that

$$u_n(c\tau_n,\tau_n)=\frac{\varepsilon_1(c)}{2}$$

for n large enough. Thus we obtain that $t_n-\tau_n\to+\infty$, as $n\to+\infty$. Therefore we can extract a subsequence such that

$$\begin{cases}\tilde{u}(x,t)=\lim_{n\to+\infty} u_n(x+c\tau_n,\tau_n+t),\\ \tilde{v}(x,t)=\lim_{n\to+\infty} v_n(x+c\tau_n,\tau_n+t),\end{cases}$$

which are well defined as a result of the global boundedness and some *a prior* estimates. The pair $(\tilde{u},\tilde{v})$ is a global in time solution of system (3.19), and $\tilde{u}(0,0)=\frac{\varepsilon_1(c)}{2}>0$. In addition, we have that

$$\tilde{u}(ct,t)\le\frac{\varepsilon_1(c)}{2},\ \text{for all } t\ge 0. \tag{3.23}$$

Notice that, when $\tilde{v}\not\equiv 0$, this entire solution immediately contradicts Lemma 3.3. Now it remains to consider the case of when $\tilde{v}\equiv 0$. Then

$$\frac{\partial\tilde{u}}{\partial t}(x,t)=d_1\left[(J_1*\tilde{u})(x,t)-\tilde{u}(x,t)\right]+r_1\tilde{u}(x,t)(1-\tilde{u}(x,t)),$$

which is the scalar nonlocal diffusion equation of KPP type. Recall that $c<\underline{s}=\min\{s_*,s^*\}\le s^*$, and that $\tilde{u}(x,0)\ge\not\equiv 0$, so, by Proposition 2.1, we have that

$$\tilde{u}(ct,t)\to 1,\ \text{as } t\to+\infty,$$

which contradicts (3.23). Thus we have completed the proof of Lemma 3.4. □

Proof of Theorem 3.1 We fix η and argue by contradiction by assuming that there exist $\{t_{n,k}\}$ and $\{x_{n,k}\}$ with $t_{n,k}\to+\infty$, as $k\to+\infty$, and that

$$x_{n,k}\in[(s+\eta)t_{n,k},(\underline{s}-\eta)t_{n,k}]$$

such that

$$u_n(x_{n,k},t_{n,k})\le\frac{1}{n} \tag{3.24}$$

for any positive integers n and k. However, applying Lemma 3.4, one has that

$$\liminf_{t\to+\infty} u_n((\underline{s}-\eta/2)t,t)\ge\varepsilon_2(\underline{s}-\eta/2).$$

In particular, we define another time sequence

$$t'_{n,k}:=\frac{x_{n,k}}{\underline{s}-\eta/2}\in[0,t_{n,k}),\quad \forall n\ge 0,\quad \lim_{k\to+\infty}t'_{n,k}=+\infty.$$

Then, applying Lemma 3.4, one has that

$$u_n(x_{n,k},t'_{n,k})\ge\frac{\varepsilon_2(\underline{s}-\eta/2)}{2}$$

for any k large enough. For each n, we choose such a large k and drop it from our notation for convenience. Then we can define that

$$\tau_n:=\sup\left\{t'_n\le t\le t_n|u_n\left(ct,t\right)\ge\frac{\min\{\varepsilon'_1(c),\varepsilon_2(\underline{s}-\eta/2)\}}{2}\right\},$$

where $\varepsilon_1'(c)$ comes from Lemma 3.3. Similarly to the proof of Lemma 3.4, we have that

$$t_n - \tau_n \to +\infty, \text{ as } n \to +\infty.$$

We consider the functions

$$\tilde{u}_n(x,t) = u(x + c\tau_n, \tau_n + t), \quad \tilde{v}_n(x,t) = v(x + c\tau_n, \tau_n + t),$$

and

$$\lim_{n\to\infty} \tilde{u}_n(x,t) := \tilde{u}(x,t), \quad \lim_{n\to\infty} \tilde{v}_n(x,t) := \tilde{v}(x,t).$$

Then the following properties can be derived:

$$\tilde{u}\,(0,0) = \frac{\min\{\varepsilon_1'(c), \varepsilon_2(\underline{s} - \eta/2)\}}{2}, \quad \tilde{u}\,(0,t) \le \frac{\min\{\varepsilon_1'(c), \varepsilon_2(\underline{s} - \eta/2)\}}{2}, \text{ for all } t \ge 0.$$

Regardless of whether $\tilde{v} \equiv 0$ or $\tilde{v} > 0$, we reach a contradiction with either the result of Proposition 2.1, or Lemma 3.3. This concludes the proof. □

4 Survival of the Predator v

In this section, we show the persistence of the predator v in the moving frames with speeds in the interval $(s, \underline{s})$, where we recall that $\underline{s} = \min\{s_*, s^*\}$.

Theorem 4.1 (Uniform spreading of v) Let $\bar{\alpha} = \max\{-\alpha(-\infty), 1\}$. Assume that (J1), (J2), (α1)–(α3), (H1) and $d_1 > r_1\bar{\alpha} + \frac{r_1 a}{2} + \frac{r_2 b(b-1)}{2}$ and $d_2 > r_2(b-1) + \frac{r_2 b(b-1)}{2} + \frac{a r_1}{2}$. Let initial data $(u_0, v_0) \in H$ with $u_0 \not\equiv 0$ and $v_0 \not\equiv 0$ be given. If $s < \underline{s}$, then, for any $\eta \in (0, (\underline{s} - s)/2)$, there exists $\varepsilon > 0$ such that $\liminf_{t\to+\infty} \inf_{(s+\eta)t \le |x| \le (\underline{s}-\eta)t} v(x,t) \ge \varepsilon$.

The method is the same as for the prey, but before proving Theorem 4.1, we give the following three lemmas:

Lemma 4.2 (Pointwise weak spreading) Assume that $s < \underline{s}$. Then, for any $c \in (s, \underline{s})$, there exists $\varepsilon_3(c) > 0$ such that, for any $(u_0, v_0) \in H$ with $u_0 \not\equiv 0$ and $v_0 \not\equiv 0$, the corresponding solution (u, v) of (1.1) satisfies that $\limsup_{t\to\infty} v(ct,t) \ge \varepsilon_3(c)$.

Proof We let $c \in (s, \underline{s})$, and assume by contradiction that there exists a sequence of solutions $\{(\tilde{u}_n, \tilde{v}_n)\}$ with initial data $\{u_{0,n}, v_{0,n}\} \subset H$, with $u_{0,n} \not\equiv 0$ and $v_{0,n} \not\equiv 0$, such that

$$\lim_{n\to+\infty} \limsup_{t\to\infty} \tilde{v}_n(ct,t) = 0. \tag{4.1}$$

Then, for each n, we can choose t_n large enough such that

$$\lim_{n\to+\infty} \sup_{t\ge t_n} \tilde{v}_n(ct,t) = 0. \tag{4.2}$$

Next we claim that, for any $R > 0$,

$$\limsup_{n\to+\infty} \{ \sup_{t\ge t_n, |x-ct|\le R} \tilde{v}_n(x,t)\} = 0. \tag{4.3}$$

Indeed, assume by contradiction that there exist sequences $t_n' \ge t_n$ and $x_n \in [ct_n' - R, ct_n' + R]$ such that

$$\liminf_{n\to+\infty} \tilde{v}_n(x_n, t_n') > 0. \tag{4.4}$$

By using arguments similar to those for Lemma 2.2, we can extract a subsequence such that the following convergence holds locally uniformly in $(x,t) \in \mathbb{R} \times \mathbb{R}$:

$$\begin{cases} \lim\limits_{n\to\infty} \tilde{u}_n\left(x+x_n, t+t_n'\right) = \tilde{u}_\infty(x,t), \\ \lim\limits_{n\to\infty} \tilde{v}_n\left(x+x_n, t+t_n'\right) = \tilde{v}_\infty(x,t). \end{cases}$$

Here the limit function $(\tilde{u}_\infty, \tilde{v}_\infty)$ is an entire solution of the system

$$\begin{cases} \dfrac{\partial \tilde{u}_\infty}{\partial t}(x,t) = d_1\left[\left(J_1 * \tilde{u}_\infty\right)(x,t) - \tilde{u}_\infty(x,t)\right] + r_1 \tilde{u}_\infty(x,t)\left[1 - \tilde{u}_\infty(x,t) - a\tilde{v}_\infty(x,t)\right], \\ \dfrac{\partial \tilde{v}_\infty}{\partial t}(x,t) = d_2\left[\left(J_2 * \tilde{v}_\infty\right)(x,t) - \tilde{v}_\infty(x,t)\right] + r_2 \tilde{v}_\infty(x,t)\left[-1 + b\tilde{u}_\infty(x,t) - \tilde{v}_\infty(x,t)\right]. \end{cases} \tag{4.5}$$

It is clear that $\tilde{v}_\infty \geq 0$. By extracting subsequences $t_n' \geq t_n$, $x_n \to x_\infty \in [ct_n' - R, ct_n' + R]$ and using (4.1), we deduce that $\tilde{v}_\infty(ct_n' - x_\infty, 0) = 0$. According to the strong maximum principle, we obtain that $\tilde{v}_\infty \equiv 0$. On the other hand, by (4.4), we deduce that $\tilde{v}_\infty(0,0) > 0$, which is a contradiction. Therefore, (4.3) holds.

Now, we claim that

$$\limsup_{n\to+\infty}\{\sup_{t\geq t_n, |x-ct|\leq R} \tilde{u}_n(x,t)\} = 1, \text{ for any } R > 0. \tag{4.6}$$

We assume by contradiction that there is a sequence $\{(x_n, t_n')\}$ with $t_n' \geq t_n$ and $x_n \in [ct_n' - R, ct_n' + R]$ such that

$$\limsup_{n\to+\infty} \tilde{u}_n(x_n, t_n') < 1.$$

By some *a prior* estimates, the Arzelà-Ascoli theorem and a diagonal extraction process, we can extract a subsequence $t_n' \to \infty$ such that

$$(\tilde{u}_n, \tilde{v}_n)(x + x_n, t + t_n') \to (u_\infty, v_\infty)(x,t) \text{ locally uniformly as } n \to \infty,$$

where $(u_\infty, v_\infty)(x,t)$ satisfies that

$$\begin{cases} \dfrac{\partial u_\infty}{\partial t}(x,t) = d_1\left[\left(J_1 * u_\infty\right)(x,t) - u_\infty(x,t)\right] + r_1 u_\infty(x,t)\left[1 - u_\infty(x,t) - av_\infty(x,t)\right], \\ \dfrac{\partial v_\infty}{\partial t}(x,t) = d_2\left[\left(J_2 * v_\infty\right)(x,t) - v_\infty(x,t)\right] + r_2 v_\infty(x,t)\left[-1 + bu_\infty(x,t) - v_\infty(x,t)\right]. \end{cases}$$

Since $v_\infty(0,t) = 0$ for all $t > 0$, by the strong maximum principle we get that $v_\infty \equiv 0$. In particular, u_∞ satisfies that

$$\frac{\partial u_\infty}{\partial t}(x,t) = d_1\left[\left(J_1 * u_\infty\right)(x,t) - u_\infty(x,t)\right] + r_1 u_\infty(x,t)\left(1 - u_\infty(x,t)\right), \quad (x,t) \in \mathbb{R}^2.$$

On the other hand, by Theorem 3.1, we have that

$$\inf_{(x,t)\in\mathbb{R}^2} u_\infty(x,t) > 0.$$

This implies that $u_\infty \equiv 1$, which is a contradiction to $u_\infty(0,0) < 1$, by our choices of x_n and t_n'. Hence (4.6) holds.

For any small $\delta_1 > 0$ and large $R > 0$, we have that

$$\frac{\partial \tilde{v}_n}{\partial t}(x,t) \geq d_2\left[\left(J_2 * \tilde{v}_n\right)(x,t) - \tilde{v}_n(x,t)\right] + r_2 \tilde{v}_n(x,t)\left(-1 + b - \delta_1\right), |x - ct_n| \leq R, t \geq t_n$$

for any n large enough. Similarly to the proof of Lemma 3.2, we infer that $\tilde{v}_n(ct,t) \to +\infty$ as $t \to +\infty$, which is a contradiction to (4.3). This completes the proof of Lemma 4.2. $\square$

Lemma 4.3 For any $c \in [0, \underline{s})$, there exists $\varepsilon_3'(c) > 0$ such that, for any initial data satisfying $(u_0, v_0) \in H$ with $u_0 \not\equiv 0$ and $v_0 \not\equiv 0$, the corresponding solution (u, v) of (3.19) satisfies that

$$\limsup_{t \to +\infty} u(ct, t) \geq \varepsilon_3'(c).$$

Proof The proof of Lemma 4.3 is similar to that of Lemma 4.2, so we omit it. □

Lemma 4.4 (Pointwise spreading) Assume that $s < \underline{s}$. Then, for any $(u_0, v_0) \in H$ with $u_0 \not\equiv 0$ and $v_0 \not\equiv 0$, for all $c \in (s, \underline{s})$, there exists $\varepsilon_4(c) > 0$ such that the corresponding solution (u, v) of (1.1) satisfies that

$$\liminf_{t \to +\infty} v(ct, t) \geq \varepsilon_4(c).$$

Proof The method is the same as that of Lemma 3.4. Fixing $c \in (s, \underline{s})$ and proceeding by contradiction, we find an entire solution (u_∞, v_∞) of (3.19) such that $v_\infty(0,0) = \frac{\varepsilon_3(c)}{2} > 0$ and

$$v_\infty(ct, t) \leq \frac{\varepsilon_3(c)}{2}, \forall t \geq 0. \tag{4.7}$$

Moreover, by Theorem 3.1, we have that

$$\inf_{(x,t) \in \mathbb{R}^2} u_\infty(x, t) > 0,$$

which implies that $u_\infty \equiv 1$. Thus,

$$\frac{\partial v_\infty}{\partial t}(x, t) = d_2 \left[(J_2 * v_\infty)(x, t) - v_\infty(x, t) \right] + r_2 v_\infty(x, t) \left[-1 + b - v_\infty(x, t) \right].$$

Since $c < \underline{s} \leq s^*$, and by Proposition 3.3 and Assumption (H1), we have that

$$\lim_{t \to +\infty} v_\infty(ct, t) = b - 1 > 0, \tag{4.8}$$

which contradicts (4.7). This completes the proof of Lemma 4.4. □

Proof of Theorem 4.1 We fix η and argue by contradiction by assuming that there exist $\{t_{n,k}\}$ and $\{x_{n,k}\}$ with $t_{n,k} \to +\infty$, as $k \to +\infty$, and that

$$x_{n,k} \in [(s + \eta) t_{n,k}, (\underline{s} - \eta) t_{n,k}],$$

such that

$$v_n(x_{n,k}, t_{n,k}) \leq \frac{1}{n}$$

for any positive integers n and k. However, applying Lemma 4.4, we have that

$$\liminf_{t \to +\infty} v_n((\underline{s} - \eta/2)t, t) \geq \varepsilon_4(\underline{s} - \eta/2).$$

In particular, we define another time sequence as

$$t_{n,k}' := \frac{x_{n,k}}{\underline{s} - \eta/2} \in [0, t_{n,k}), \quad \forall n \geq 0, \quad \lim_{k \to +\infty} t_{n,k}' = +\infty.$$

Then, applying Lemma 4.4, one has that

$$v_n(x_{n,k}, t_{n,k}') \geq \frac{\varepsilon_4(\underline{s} - \eta/2)}{2}$$

for any k large enough. For each n, we choose such a large k and drop it from our notation for convenience. Then we can define that

$$\tau_n := \sup \left\{ t_n' \leq t \leq t_n | v_n(ct, t) \geq \frac{\min\{\varepsilon_3'(c), \varepsilon_4(\underline{s} - \eta/2)\}}{2} \right\},$$

where $\varepsilon_3'(c)$ comes from Lemma 4.3. Similarly to the proof of Lemma 3.4, we have that

$$t_n - \tau_n \to +\infty, \text{ as } n \to +\infty.$$

We consider the functions

$$\tilde{u}_n(x,t) = u(x + c\tau_n, \tau_n + t), \quad \tilde{v}_n(x,t) = v(x + c\tau_n, \tau_n + t), \tag{4.9}$$

and

$$\lim_{n\to\infty} \tilde{u}_n(x,t) := \tilde{u}(x,t), \quad \lim_{n\to\infty} \tilde{v}_n(x,t) := \tilde{v}(x,t).$$

Then we deduce that

$$\tilde{v}\,(0,0) = \frac{\min\{\varepsilon_3'(c), \varepsilon_4(\underline{s} - \eta/2)\}}{2}, \quad \tilde{v}\,(0,t) \le \frac{\min\{\varepsilon_3'(c), \varepsilon_4(\underline{s} - \eta/2)\}}{2}, \text{ for all } t \ge 0.$$

Regardless of whether $\tilde{v} \equiv 0$ or $\tilde{v} > 0$, we reach a contradiction with either the result of Proposition 2.1, or Lemma 4.3. This completes the proof of Theorem 4.1. □

Conflict of Interest The authors declare no conflict of interest.

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